

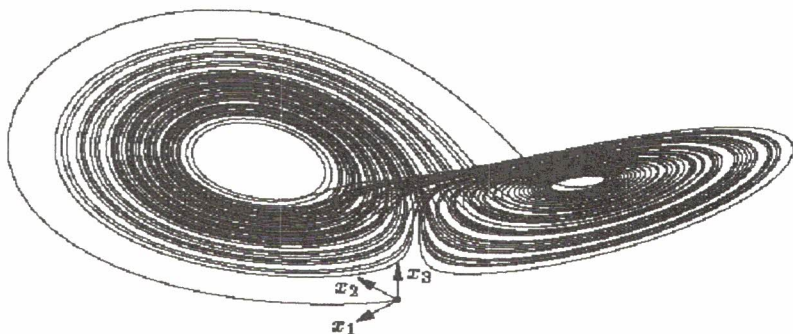
**Ministry of Education and Science of Ukraine**

**Odessa State Environmental University**

**O.I. Gerasymov**

**Lecture Notes**

*Introduction to nonlinear analysis for  
radioecologists*



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*Друкується за рішенням Вченої ради Одеського державного екологічного університету (протокол № 8 від 31.10.2013 р.).*

**Г 37 Герасимов, Олег Іванович**

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Конспект лекцій з елементів нелінійного аналізу спрямовано на порозуміння формулювань і розв'язання типових задач навколишнього середовища (зокрема - екології) та інших междисциплінарних напрямів підготовки ВНЗ, як фундаментальних, так і прикладних (інженерних).

Конспект лекцій пропонується для магістрів за спеціальністю радіоекологія, містить найбільш загальні формулювання та моделі в області нелінійних задач, які складають елементну базу дисципліни.

Proposed lecture notes in Nonlinear analysis include introductory elements of the subject and properly directed to magisters who specialized in Radioecology. Selected topics and widely known models which has been outlined could be practically useful also for representatives from other interdisciplinary specializations.

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## 1. LECTURE MODULE 1

### 1.1 INTRODUCTION

In a linear world, the effects being always proportional to their causes, everything would be “simple” since the tools that would allow us to represent it as a superposition of elementary states are rather well mastered.

Unfortunately the world is “complex.” As a matter of fact, we often observe effects which saturate in spite of an increase of their causes, or which go in different and somehow unexpected ways. All this is ascribed to nonlinearities.

Nonlinear phenomena are encountered in all areas of the quantitative sciences. Today it is becoming clearer that the linear world, for which mathematics has been developed over a period of 300 years is but a tiny corner of a far richer world which is being unravelled slowly.

A few examples are: nonlinear mechanical vibrations, population dynamics (e.g., evolution of predator-prey populations), electronic circuits (e.g., triodes), laser physics, astrophysics (e.g., planetary motion), the heart-beat, biological clocks, war games, nonlinear diffusion: fluid dynamics, meteorology, plasma physics, chemical reactions in solutions, Stefan problem (diffusion in domains with moving boundaries), nonlinear wave equations, time-delay process, critical phenomena (phase transitions).

Already in the 18th century, Clairot, Lagrange and Laplace developed perturbation methods for the treatment of nonlinear problems in celestial mechanics. Jacobi, Poincare and Lindstedt studied similar problems towards the end of the 19th century. Quite a few concepts that have evolved in recent years can be traced back to the work of Poincaré. Towards the end of the 19th century Rayleigh studied self sustained vibrations in his work on the theory of sound. His equation for the self sustaining oscillations in organ pipes is equivalent to the equation developed by Van der Pol in the 1920's for

the current in a triode. Both equations yield anharmonic self-sustaining oscillations.

During the 1930's to 1940's Krylov and Bogoliubov developed methods for analyzing small nonlinear perturbations. Later on, interest arose in nonlinear problems in which the nonlinearity is not small. Important milestones were (among others) the Lorenz equations as a model for fluid motion, Feigenbaum's analysis of discrete nonlinear maps and their relevance to the behavior of Nonlinear Dynamics continuous nonlinear systems, and the advent of the concepts of chaos and fractals.

## 1.2 DYNAMICAL SYSTEMS

### General derivations

Newtonian mechanics is the archetype of deterministic dynamical theories.

Governed by an equation of the form

$$m\ddot{X} = F,$$

where  $X$  represents the position of a particle with mass  $m$  submitted to a force  $F$ , it accounts for processes that are invariants under a change of the arrow of time.

A traditional example of linear system is the harmonic oscillator that describes the motion of a mass attached to an ideal spring with a restoring force proportional to the extension  $F = -kX$  (Hooke's law). This elastic force, of internal origin, derives from a potential:

$$F = -\frac{\partial V}{\partial X} \quad \text{with} \quad v = \frac{1}{2}kX^2 \quad (1.1)$$

Here we have

$$m\ddot{X} + kX = 0 \quad (1.2)$$

The evolution is uniquely determined when initial conditions are specified:

$$X = X^{(0)} \quad \text{and} \quad \dot{X} = \dot{X}^{(0)} \quad \text{at} \quad t = 0 \quad (1.3)$$

The solution then reads:

$$X(t) = X^{(0)}\text{Cos}(\omega_0 t) + (\dot{X}^{(0)}/\omega_0)\text{Sin}(\omega_0 t) \quad \text{with} \quad \omega_0^2 = k/m \quad (1.4)$$

Right now, it turns out advantageous to substitute a geometrical description (Fig. 1.1) to this analytical description. So, let us consider trajectories in the phase space where coordinates are the position  $X$  and the momentum

$$P = m\dot{X} = m\dot{X} \quad (1.5)$$

Equation of linear oscillator now reads

$$\dot{P} = -kX$$

(1.6)

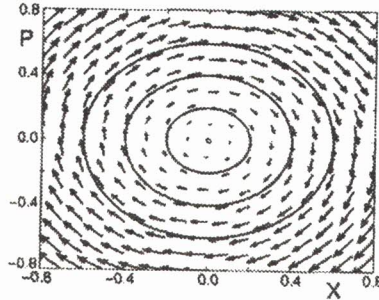


Fig. 1.1 Vector field and trajectories in the phase plane, the plane of variables  $X_1 = X$  and  $X_2 = P$ .

One can trace back the origin of this representation from the need to turn a high order differential equation into a first order differential system. On general grounds, this is done by introducing the successive derivatives as intermediate variables. For example, equation:

$$d^n X/dt^n = F(d^{n-1} X/dt^{n-1} X, \dots, \ddot{X}, \dot{X}, X; t) \quad (1.7)$$

is reduced to a first order system by setting

$$X_1 \equiv X, X_2 \equiv \dot{X}, \dots, X_n \equiv d^{n-1} X/dt^{n-1} \quad (1.8)$$

which yields:

$$\dot{X}_1 = X_2, \dot{X}_2 = X_3, \dots, \dot{X}_n = F(X_n, \dots, X_3, X_2, X_1; t). \quad (1.9)$$

This operation is preliminary to any numerical implementation in view of simulation.

There is a slight disadvantage to start from mechanics' for an introduction of basic concepts of the theory of dynamical systems since Newton's equations endow the phase space with a specific structure and give it an even number of dimensions,  $d = 2n$ . As seen above, this space is indeed constructed as the product of the configuration space (variables  $X_i$ ,  $i = 1, \dots, n$ ) and of the momentum space (variables  $P_i$ ,  $i = 1, \dots, n$ ), each pair

of conjugate variables  $(X_i, P_i)$  forming a degree of freedom. In the general case when the state of a system is specified using a supposedly sufficient number of variables, no longer grouped by pairs, we will call any of these state variable a 'degree of freedom'. Accordingly, the 'phase space' will then be the space where these variables "live" and the dimension of this space will just be the number of these variables. In agreement with the intuitive concept of determinism, the dimension  $d$  is also exactly the number of conditions necessary to specify any evolution uniquely. A way to avoid ambiguities would be to keep the terms 'phase space' and 'degrees of freedom' to mechanics in a strict sense and to use state variables and state space in all other cases, which we will not do since the risk of confusion is limited.

Anyway, let us consider the general case of a phase space  $X$  with dimension  $d$  spanned by  $d$  variables  $\{X_1, \dots, X_d\} \equiv X$ . The evolution of the so-defined variables is governed by a system that symbolically reads

$$\dot{X} = \mathcal{F}(X; t), \tag{1.10}$$

where  $\mathcal{F}$  is a set of  $d$  functions representing the components of a vector field defined on  $X$  that specifies the "velocity" of the point representing the system in its phase space.

When the properties of  $\mathcal{F}$  guarantee the existence and uniqueness of the solution to the initial value problem, in practice when the vector field  $\mathcal{F}$  is differentiable with continuous first derivative, one says that it defines a flow on phase space.

When  $t$  is explicitly absent from the definition of  $\mathcal{F}$ , the system is said to be autonomous, otherwise it is forced. In practice, among forced systems, only periodically forced systems will be of interest to us, i.e. systems such that

$\mathcal{F}(X; t + T) \equiv \mathcal{F}(X; t)$  for some minimal time interval  $T$  called the period. Within this class one often distinguishes parametric forcing for



which the expression of  $\mathcal{F}$  changes in time, e.g. the parametric linearoscillator (Mathieu equation):

$$\ddot{X} + (1 + a \sin(\omega t))X = 0, \quad T = 2\pi/\omega, \quad (1.11)$$

from external forcing where an otherwise autonomous system is submitted to a periodic force independent of its state, i.e.

$$\ddot{X} + X = f \sin(\omega t). \quad (1.12)$$

Noisy systems can be understood as particular forced systems with a random forcing. In case of additive noise, this defines the so-called Langevin equation

$$\dot{X} = \mathcal{F}(X) + \Xi(t), \quad (1.13)$$

where  $\Xi(t)$  is a random vector function. Quite different tools of statistical essence are then required which will not be introduced here since we want to stick to the deterministic point of view.

During its evolution, the system follows a phase space trajectory starting at  $X^{(0)}$  when  $t = t^{(0)}$  and obtained by integration of evolution equation:

$$X(t) = X^{(0)} + \int_{t^{(0)}}^t \mathcal{F}(X(t'); t') dt'. \quad (1.14)$$

The orbit is the set of points in  $X$  visited by the system in the course of a given trajectory. The description of a system's dynamics in terms of sets of orbits, called its phase portrait, as a function of its control parameters, is the field of qualitative dynamics.

### Formalism of analytical mechanics

In order to ease the solution to some exercises, we give here an introduction to the analytical formalism of classical mechanics that allows one

to pass from the Newton equations (second order in time) to the Hamilton equations (first order in time) giving to the intermediate variables so-introduced their status of conjugate momenta to the generalized coordinates.

For a system of  $n_N$  (subscript 'N' for Newton) material points with masses  $m_i$  and positions  $\mathbf{X}_i \equiv (x_i, y_i, z_i)$  submitted to forces  $\mathbf{f}_i$  deriving from a potential  $\mathcal{V}(\{\mathbf{X}_i\}; t)$ , the Newton equations read:

$$m_i \ddot{\mathbf{X}}_i = \mathbf{f}_i = -\frac{\partial \mathcal{V}}{\partial \mathbf{X}_i}, \quad (i = 1, \dots, n_N). \quad (1.15)$$

The kinetic energy is defined by:

$$\mathcal{T} = \frac{1}{2} \sum_{i=1}^{n_N} m_i (\dot{\mathbf{X}}_i)^2, \quad (1.16)$$

and one checks that the total energy

$$E = \mathcal{T} + \mathcal{V} \quad (1.17)$$

is conserved. But this so-called Newtonian formulation makes the treatment of systems with constraints somehow awkward. The Lagrangian formalism answers this problem by introducing  $n$  ('L' for Lagrange) generalized coordinate.

Let

$$\mathbf{q} \equiv \{q_j; j = 1, \dots, n_L\}, \quad q_j = q_j(\{\mathbf{X}_i; i = 1, \dots, n_N\}; t),$$

be the change of variables from the  $\mathbf{X}_i$ s to the  $q_j$ s expressing the constraints. The Lagrangian is then defined by

$$\mathcal{L} = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T} - \mathcal{V}, \quad (1.18)$$

so that Newton equations can be rewritten as a set of Lagrange equations:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\dot{q}_j)} - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \quad j = 1, \dots, n_L. \quad (1.19)$$

At this stage, the system is still governed by second order differential equations. One then defines the momenta  $p \equiv \{p_j\}$  conjugated to the coordinates  $q \equiv \{q_j\}$  by:

$$p_j \equiv \frac{\partial \mathcal{L}}{\partial (\dot{q}_j)}, \quad (1.20)$$

and the Hamiltonian by:

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \sum_j p_j \dot{q}_j - \mathcal{L}. \quad (1.21)$$

The Lagrange equations then turn into the Hamilton equations:

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad j = 1, \dots, n_L. \quad (1.22)$$

The phase space  $\mathbf{X}$  is the Cartesian product of the configuration space with coordinates  $q_j$  and the momentum space with coordinates  $p_j$ , thus with dimension  $d = 2n_L$ . The vector field governing the dynamics then reads

$$\mathcal{F}_{q_j} \equiv \frac{\partial \mathcal{H}}{\partial p_j} \quad \text{and} \quad \mathcal{F}_{p_j} \equiv -\frac{\partial \mathcal{H}}{\partial q_j}, \quad (1.23)$$

expressions that provide it with a so-called 'symplectic' structure insuring the conservation of phase space volumes automatically ( $\text{div } \mathcal{F} = 0$ , the Liouville theorem).

### Phase plane

The general form of a linear second-order system is

$$\dot{x} = ax_1 + bx_2 \quad (1.24a)$$

$$x_2 = cx_1 + dx_2 \quad (1.24b)$$

Transform these equations into a scalar second-order

differential equation in the form  $b\ddot{x}_2 = bcx_1 + d(\dot{x}_1 - ax_1)$ . Consequently, differentiation of (1.24a) and then substitution of (1.24b) leads to  $\ddot{x}_1 = (a+d)\dot{x}_1 + (cb-ad)x_1$ . Therefore, we will simply consider the second-order linear system described by

$$\ddot{x} + a\dot{x} = bx = 0 \tag{1.25}$$

To obtain the phase portrait of this linear system, we solve for the time history

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \text{ for } \lambda_1 \neq \lambda_2 \tag{1.26a}$$

$$x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \text{ for } \lambda_1 = \lambda_2 \tag{1.26b}$$

where the constant  $\lambda_1, \lambda_2$  are the solutions of the characteristic equation

$$s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0 \tag{1.27}$$

The roots  $\lambda_1, \lambda_2$  can be explicitly represented as

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \text{ and } \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} \tag{1.28}$$

For linear systems described by (1.25), there is only one singular point ( $b \neq 0$ ), namely the origin. However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of  $a$  and  $b$ . The following cases can occur

$\lambda_1, \lambda_2$  are both real and have the same sign (+ or -)

$\lambda_1, \lambda_2$  are both real and have opposite sign

$\lambda_1, \lambda_2$  are complex conjugates with non-zero real parts

$\lambda_1, \lambda_2$  are complex conjugates with real parts equal to 0

*Stable or unstable node* (Fig. 1.1.a -b)

The first case corresponds to a node. A node can be stable or unstable:

$\lambda_1, \lambda_2 < 0$  : singularity point is called *stable node*.

$\lambda_1, \lambda_2 > 0$  : singularity point is called *unstable node*.

There is no oscillation in the trajectories.

*Saddle point* (Fig. 1.1.c)

The second case ( $\lambda_1 < 0 < \lambda_2$ ) corresponds to a *saddle point*. Because trajectories diverge the unstable to infinity. pole  $\lambda_2$ , almost all of the system trajectories diverge to infinity.

*Stable or unstable locus* (Fig. 1.1.d-e)

The third case corresponds to a focus.

$\text{Re}(\lambda_1, \lambda_2) < 0$  : *stable focus*

$\text{Re}(\lambda_1, \lambda_2) > 0$  : *unstable focus*

*Center point* (Fig. 1.1.f)

The last case corresponds to a *certain point*. All trajectories are ellipses and the singularity point is the centre of these ellipses.

*Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.*

### **Phase Plane Analysis of Nonlinear Systems**

In discussing the phase plane analysis of nonlinear system, two points should be kept in mind:

Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of nonlinear systems can be approximated by the behavior of a linear system.

Nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles.

If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, we can shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Eq.(2.1) with a singular point at 0. Using Taylor expansion, Eqs. (2.1) can be rewritten in the form

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 + g_1(x_1, x_2) \\ \dot{x}_2 &= cx_1 + dx_2 + g_2(x_1, x_2)\end{aligned}\tag{1.29}$$

where  $g_1, g_2$  contain higher order terms.

In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2\end{aligned}\tag{1.30}$$

As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Fig. 1.1.

In the phase plane, a *limit cycle is defined as an isolated closed curve*. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with near by trajectories converging or diverging from it).

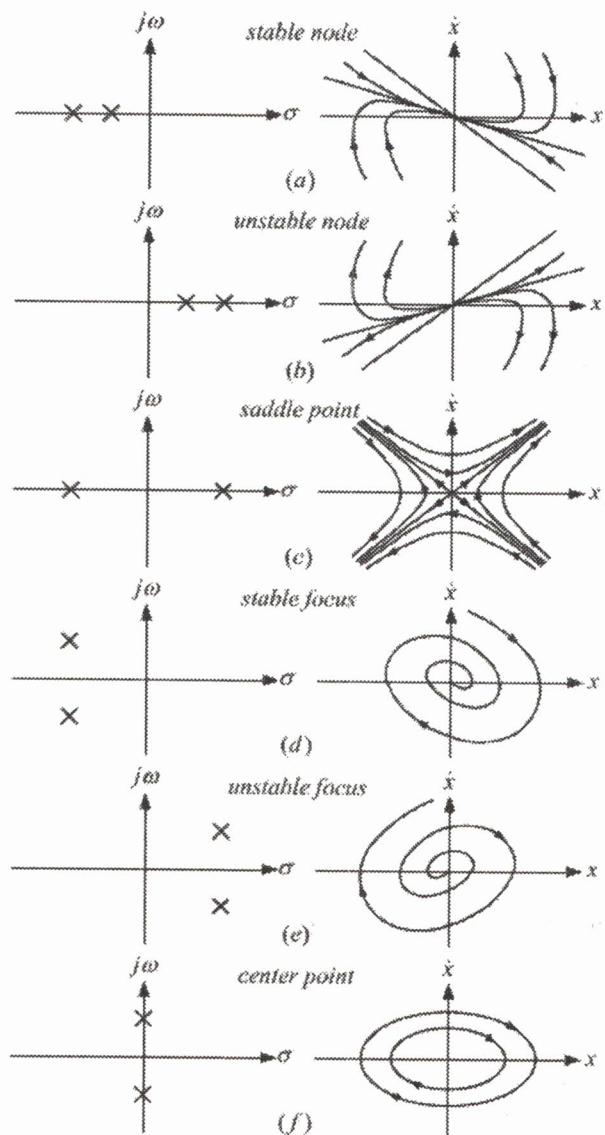


Fig 1.1 Phase-portraits of linear systems

In the phase plane, a *limit cycle* is defined as an isolated closed curve. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with near by trajectories converging or diverging from it).

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, we can distinguish three kinds of limit cycles.

**Stable Limit Cycles:** all trajectories in the vicinity of the limit cycle converge to it as  $t \rightarrow \infty$  (Fig. 1.2.a).

**Unstable Limit Cycles:** all trajectories in the vicinity of the limit cycle diverge to it as  $t \rightarrow \infty$  (Fig. 1.2.b)

**Semi-Stable Limit Cycles:** some of the trajectories in the vicinity of the limit cycle converge to it as  $t \rightarrow \infty$  (Fig. 1.2.c)

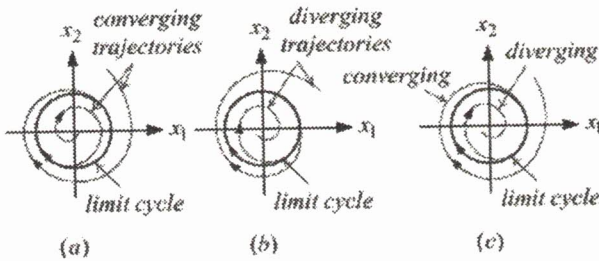


Fig. 1.2 Stable, unstable, and semi-stable limit cycles

Consider the following nonlinear systems

$$(a) \begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases} \quad (1.31)$$

$$(b) \begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases} \quad (1.32)$$

$$(c) \begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{cases} \quad (1.33)$$

By introducing a polar coordinates



$$r = \sqrt{x_1^2 + x_2^2} \quad \Theta(t) = \tan^{-1} \left( \frac{x_2}{x_1} \right) \quad (1.34)$$

the dynamics of (1.31) are transformed as

$$\frac{dr}{dt} = -r(r^2 - 1) \quad \frac{d\Theta}{dt} = -1 \quad (1.35)$$

When the state starts on the unicycle, the above equation shows that  $\dot{r}(t)=0$ . Therefore, the state will circle around the origin with a period  $1/2\pi$ . When  $r < 1$ , then  $\dot{r} > 0$ . This implies that the state tends to the circle from inside. When  $r > 1$ , then  $\dot{r} < 0$ . This implies that the states tend to the unit circle from outside. Therefore, the unit circle is a stable limit cycle. This can also be concluded by examining the analytical solution of (1.31)

$$r(t) = \frac{1}{\sqrt{1 + c_0 e^{-2t}}} \text{ and, } \Theta(t) = \Theta_0 - t \text{ where } c_0 = \frac{1}{r_0^2} - 1 \quad (1.36)$$

Similarly, we can find that the system (b) has an unstable limit cycle and system (c) has a semi-stable limit cycle.

### Strange Attractor

Luckily, in many cases, despite the tremendous sensitivity to slight changes in initial conditions or control parameters, the evolution of nonlinear systems does exhibit some regularities. Following the motion of the system in (the multi-dimensional) phase space, one sometimes finds that the "erratic" evolution is attracted asymptotically in time onto a subspace of a lower dimension. For example, if phase space is three-dimensional, then that subspace could be a two dimensional surface. On that subspace the "world line" describing the evolution of our dynamical system still exhibits the same features of high sensitivity to changes in initial conditions or control parameters and chaotic behavior. In addition, the collection of all the points through which the world line of the system passes is a strange creature. While they may be everywhere dense in that subspace (or a portion thereof) they still are of a lower "dimension" than that of the subspace. In fact, in some

geometrical terms they are considered to have fractional dimension. (For instance, for a three-dimensional system attracted onto a two-dimensional subspace, the collection of all the points on that surface, through which the system passes may have a dimension less than 2 and more than 1 - it is "more" than a line, and "less" than a surface). Such an attracting set of points is called a strange attractor.

This is the saving grace for computers. While they are unable to predict numerically the precise trajectories of many nonlinear systems in phase space, the "incorrect" solutions they produce are attracted to the same strange attractors as the real systems do. Thus, a graphic representation of the evolution of a given nonlinear system in phase space based on a numerical solution may reveal to us the existence of strange attractors.

An attractor of a dynamical system  $\dot{\varphi} = V(\varphi)$  is the set of  $\varphi$  values that the system evolves to after a sufficiently long time. For  $N = 1$  the only possible attractors are stable fixedpoints. For  $N = 2$ , we have stable nodes and spirals, but also stable limit cycles. For  $N > 2$  the situation is qualitatively different, and a fundamentally new type of set, the strange attractor, emerges.

### **The Lorenz Model**

The canonical example of an  $N = 3$  strange attractor is found in the Lorenz model. E.N. Lorenz, in a seminal paper from the early 1960's, reduced the essential physics of the coupled partial differential equations describing Rayleigh-Benard convection (a fluid slab of finite thickness, heated from below - in Lorenz's case a model of the atmosphere warmed by the ocean) to a set of twelve coupled nonlinear ordinary differential equations. Lorenz's intuition was that his weather model should exhibit recognizable patterns over time. What he found instead was that in some cases, changing his initial conditions by a part in a thousand rapidly led to totally different behavior. This sensitive dependence on initial conditions is a hallmark of chaotic systems.

The essential physics (or mathematics?) of Lorenz's  $N = 12$  system is elicited by the reduced  $N = 3$  system,

$$\begin{aligned}\dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= XY - bZ,\end{aligned}\tag{1.37}$$

where  $\sigma$ ,  $r$ , and  $b$  are all real and positive. Here  $t$  is the familiar time variable (appropriately scaled), and  $(X, Y, Z)$  represent linear combinations of physical fields, such as global wind current and poleward temperature gradient. These equations possess a symmetry under  $(X, Y, Z) \rightarrow (-X, -Y, Z)$ , but what is most important is the presence of nonlinearities in the second and third equations.

The Lorenz system is dissipative because phase space volumes contract:

$$\nabla \cdot \mathbf{V} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\sigma + b + 1).\tag{1.38}$$

Thus, volumes contract under the flow. Another property is the following.

Let

$$F(X, Y, Z) = \frac{1}{2}X^2 + \frac{1}{2}Y^2 + \frac{1}{2}(Z - r - \sigma)^2.\tag{1.39}$$

Then

$$\begin{aligned}\dot{F} &= X\dot{X} + Y\dot{Y} + (Z - r - \sigma)\dot{Z} \\ &= -\sigma X^2 - Y^2 - b\left(Z - \frac{1}{2}r - \frac{1}{2}\sigma\right)^2 + \frac{1}{4}b(r + \sigma)^2.\end{aligned}\tag{1.40}$$

Thus,  $\dot{F} < 0$  outside an ellipsoid, which means that all solutions must remain bounded in phase space for all times.

Let us show example of phase trajectories evolution of Lorenz model

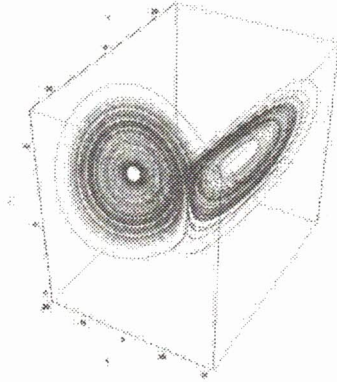


Fig. 1.3: Evolution of the Lorenz equations for  $\sigma = 10$ ,  $b = 8/3$ , and  $r = 8$ , with initial conditions  $(X_0, Y_0, Z_0) = (0, 1, 0)$ , showing the 'strange attractor'.

## 1.3 STABILITY ANALYSIS

### Stability of nonlinear dynamical systems

Often, one is interested in the stability of solutions of the dynamical equations. For example, if a perturbation method is employed, it is desirable to know for how long in time the approximation is close to the exact, but unknown, solution. In other problems one is interested in finding whether the solution of a given problem does or does not approach some asymptotic form. If it does, the rate of approach may be of interest.

Let  $\varphi(t)$  be a solution of the equation

$$\frac{dx}{dt} = f(t, x; \varepsilon) \quad x(0) = x_0 \quad (1.41)$$

### Lyapunov stability

The first question that arises is the degree of sensitivity of the solution to a slight change in the initial condition. Let  $\varepsilon > 0$  be an allowed deviation in the solution, and  $\delta > 0$  - the allowable range of variation in the initial condition. If, for any arbitrarily small  $\varepsilon$ , there exists a  $\delta$  such that when the initial condition  $\mathbf{x}_0$  is replaced by  $\mathbf{x}'_0$  satisfying

$$|\mathbf{x}_0 - \mathbf{x}'_0| \leq \delta \quad (1.42)$$

the solution  $\xi(t)$  emanating from the new initial condition satisfies

$$\|\xi(t) - \varphi(t)\| < \varepsilon \quad (1.43)$$

$\varphi(t)$  is called stable in the Lyapunov sense. This is shown in Fig. 1.4.

Example: a harmonic oscillator

Consider

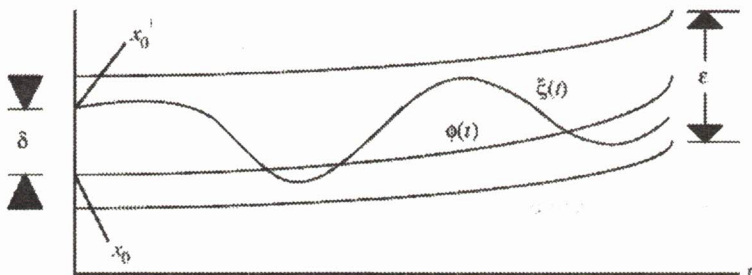


Fig. 1.4 Illustration of stability in the Lyapunov sense

$$\frac{d^2x}{dt^2} = -x \quad (1.44)$$

which can be transformed into two first order equations:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.45)$$

Define

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

then  $z=(0,0)$  is clearly a solution. Assume that the initial condition for the problem,  $z_0=(x_0, y_0)$

satisfies

$$\|z_0\| = |x_0| + |y_0| \leq \delta \Rightarrow |x_0| \leq \delta \quad |y_0| \leq \delta \quad (1.46)$$

This equation is solved by

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = \exp \left( - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) \\ y_0 \cos(t) + x_0 \sin(t) \end{pmatrix} \quad (1.47)$$

which satisfies

$$\|z\| = |x| + |y| \leq 2(|x_0| + |y_0|) \leq 2\delta \quad (1.48)$$

Thus, it is sufficient to choose  $\delta=(\varepsilon/2)$  in order to satisfy the Liapounov stability condition. The intuitive meaning is that if one chooses an initial condition which is within a certain circle of a small radius from  $z=0$ , then the solution always stays within the same circle. However, it does not tend to zero necessarily.

## 1.4. ERGODICITY

### Evolution of Phase Space Volumes.

Consider a general dynamical system,

$$\frac{d\varphi}{dt} = V(\varphi) \tag{1.49}$$

where  $\varphi(t)$  is a point in an  $n$ -dimensional phase space. Consider now a compact region  $R_0$  in phase space, and consider its evolution under the dynamics. That is,  $R_0$  consists of a set of points  $\{\varphi \mid \varphi \in R_0\}$ , and if we regard each  $\varphi \in R_0$  as an initial condition, we can define the time-dependent set  $R(t)$  as the set of points  $\varphi(t)$  that were in  $R_0$  at time  $t = 0$ :

$$R(t) = \{\varphi(t) \mid \varphi(0) \in R_0\} \tag{1.50}$$

Now consider the volume  $\Omega(t)$  of the set  $R(t)$ . We have

$$\Omega(t) = \int_{R(t)} d\mu \tag{1.51}$$

where

$$d\mu = d\varphi_1 d\varphi_2 \dots d\varphi_n \tag{1.52}$$

for an  $n$ -dimensional phase space. We then have

$$\Omega(t + dt) = \int_{R(t+dt)} d\mu' = \int_{R(t)} d\mu \left| \frac{\partial \varphi_i(t + dt)}{\partial \varphi_j(t)} \right| \tag{1.53}$$

where

$$\left| \frac{\partial \varphi_i(t + dt)}{\partial \varphi_j(t)} \right| \equiv \frac{\partial(\varphi'_1, \dots, \varphi'_n)}{\partial(\varphi_1, \dots, \varphi_n)} \tag{1.54}$$

is a determinant, which is the Jacobean of the transformation from the set of coordinates  $\{\varphi_i = \varphi_i(t)\}$  to the coordinates  $\{\varphi'_i = \varphi_i(t + dt)\}$ . But according to the dynamics, we have

$$\varphi_i(t + dt) = \varphi_i(t) + V_i(\varphi(t))dt + O(dt^2) \tag{1.55}$$



and therefore

$$\frac{\partial \varphi_i(t+dt)}{\partial \varphi_j(t)} = \delta_{ij} + \frac{\partial V_i}{\partial \varphi_j} dt + O(dt^2) \quad (1.56)$$

We now make use of the equality

$$\ln \det M = \text{Tr} \ln M \quad (1.57)$$

for any matrix M, which gives us, for small  $\epsilon$ ,

$$\det(1 + \epsilon A) = \exp \text{Tr} \ln(1 + \epsilon A) = 1 + \epsilon \text{Tr} A + \frac{1}{2} \epsilon^2 (\text{Tr} A)^2 - \text{Tr}(A^2) + \dots \quad (1.58)$$

Thus,

$$\Omega(t+dt) = \Omega(t) + \int_{\mathfrak{R}(t)} d\mu \nabla \cdot V dt + O(dt^2) \quad (1.59)$$

which says

$$\frac{d\Omega}{dt} = \int_{\mathfrak{R}(t)} d\mu \nabla \cdot V = \int_{\partial \mathfrak{R}(t)} dS \hat{n} \cdot V \quad (1.60)$$

Here, the divergence is the *phase space divergence*,

$$\nabla \cdot V = \sum_{i=1}^n \frac{\partial V_i}{\partial \varphi_i}, \quad (1.61)$$

and we have used Stokes' theorem to convert the volume integral of the divergence to a surface integral of  $\hat{n} \cdot V$ , where  $\hat{n}$  is the surface normal and  $dS$  is the differential element of surface area, and  $\partial R$  denotes the boundary of the region R. We see that if  $\nabla \cdot V = 0$  everywhere in phase space, then  $\Omega(t)$  is a constant, and phase space volumes are preserved by the evolution of the system.

For an alternative derivation, consider a function  $\rho(\varphi, t)$  which is defined to be the density of some collection of points in phase space at phase space position  $\varphi$  and time  $t$ . This must satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \quad (1.62)$$

This is called the *continuity equation*. It says that 'nobody gets lost'. If we integrate it over

a region of phase space R, we have

$$\frac{d}{dt} \int_{\mathfrak{R}} d\mu \rho = - \int_{\mathfrak{R}} d\mu \nabla \cdot (\rho V) = - \int_{\partial \mathfrak{R}} dS \hat{n} \cdot (\rho V) \quad (1.63)$$

It is perhaps helpful to think of  $\rho$  as a charge density, in which case  $J = \rho V$  is the current density. The above equation then says

$$\frac{dQ_{\mathfrak{R}}}{dt} = - \int_{\partial \mathfrak{R}} dS \hat{n} \cdot J \quad (1.64)$$

where  $Q_{\mathfrak{R}}$  is the total charge contained inside the region R. In other words, the rate of increase or decrease of the charge within the region R is equal to the total integrated current flowing in or out of R at its boundary.

The Leibniz rule lets us write the continuity equation as

$$\frac{\partial \rho}{\partial t} + V \cdot \nabla_{\rho} + \rho \nabla \cdot V = 0 \quad (1.65)$$

But now suppose that the phase flow is divergenceless, i.e.  $\nabla \cdot V = 0$ . Then we have

$$\frac{D\rho}{Dt} \equiv \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \rho = 0 \quad (1.66)$$

The combination inside the brackets above is known as the *convective derivative*. It tells us the total rate of change of  $\rho$  for an observer *moving with the phase flow*. That is

$$\frac{d}{dt} \rho(\varphi(t), t) = \frac{\partial \rho}{\partial \varphi_i} \frac{d\varphi_i}{dt} + \frac{\partial \rho}{\partial t} = \sum_{i=1}^n V_i \frac{\partial \rho}{\partial \varphi_i} + \frac{\partial \rho}{\partial t} = \frac{D\rho}{Dt} \quad (1.67)$$

If  $\frac{D\rho}{Dt} = 0$ , the local density remains the same during the evolution of the system. If we consider the ‘characteristic function’

$$\rho(\varphi, t = 0) = \begin{cases} 1 & \varphi \in \mathfrak{R}_0 \\ 0 & \text{otherwise} \end{cases} \quad (1.68)$$

then the vanishing of the convective derivative means that the image of the set  $\mathfrak{R}_0$  under time evolution will always have the same volume.

Hamiltonian evolution in classical mechanics is volume preserving. The equations of motion are

$$\dot{q}_i = +\frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (1.69)$$

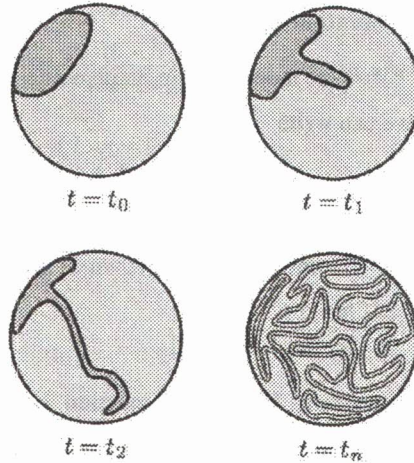


Fig 1.5. Time evolution of two immiscible fluids. The local density remains constant

A point in phase space is specified by  $r$  positions  $q_i$  and  $r$  momenta  $p_i$ , hence the dimension of phase space is  $n = 2r$ :

$$\varphi = \begin{pmatrix} q \\ p \end{pmatrix}, \quad V = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial H / \partial p \\ -\partial H / \partial q \end{pmatrix} \quad (1.70)$$

Hamilton's equations of motion guarantee that the phase space flow is divergenceless:

$$\nabla \cdot V = \sum_{i=1}^r \left\{ \frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} \right\} = \sum_{i=0}^r \left\{ \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) \right\} = 0 \quad (1.71)$$

Thus, we have that the convective derivative vanishes, viz.

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + V \cdot \nabla \rho = 0 \quad (1.72)$$

for any distribution  $\rho(\varphi, t)$  on phase space. Thus, the value of the density  $\rho(\varphi(t), t)$  is constant, which tells us that the phase flow is *incompressible*. In particular, phase space volumes are preserved.

### Liouville's equation

Let  $\rho(\varphi) = \rho(q, p)$  be a distribution on phase space. Assuming the evolution is Hamiltonian, we can write

$$\frac{\partial \rho}{\partial t} = -\dot{\varphi} \cdot \nabla \rho = -\sum_{k=1}^r \left( \dot{q}_k \frac{\partial}{\partial q_k} + \dot{p}_k \frac{\partial}{\partial p_k} \right) \rho \quad (1.73)$$

where  $\hat{L}$  is a differential operator known as the Liouvillian:

$$\hat{L} = -i \sum_{k=1}^r \left\{ \frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} \right\} \quad (1.74)$$

Eqn. 6.75, known as Liouville's equation, bears an obvious resemblance to the Schrödinger equation from quantum mechanics.

Suppose that  $\Lambda_a(\varphi)$  is conserved by the dynamics of the system. Typical conserved quantities include the components of the total linear momentum (if there is translational invariance), the components of the total angular momentum (if there is rotational invariance), and the Hamiltonian itself (if the Lagrangian is not explicitly time-dependent). Now consider a distribution  $\rho(\varphi, t) = \rho(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  which is a function only of these various conserved quantities. Then from the chain rule, we have

$$\dot{\varphi} \cdot \nabla_{\rho} = \sum_a \frac{\partial \rho}{\partial \Lambda_a} \dot{\varphi} \cdot \nabla \Lambda_a = 0 \quad (1.75)$$

since for each  $a$  we have

$$\frac{d\Lambda_a}{dt} = \sum_{\sigma=1}^r \left( \frac{\partial \Lambda_a}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial \Lambda_a}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) = \dot{\varphi} \cdot \nabla \Lambda_a = 0 \quad (1.76)$$

We conclude that any distribution  $\rho(\varphi, t) = \rho(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  which is a function solely of conserved dynamical quantities is a stationary solution to Liouville's equation.

Clearly the microcanonical distribution,

$$\rho_E(\varphi) = \frac{\delta(E - H(\varphi))}{\sum(E)} = \frac{\delta(E - H(\varphi))}{\int d\mu \delta(E - H(\varphi))}, \quad (1.77)$$

is a fixed point solution of Liouville's equation.

## 2. LECTURE MODULE 2

### 2.1 NONLINEAR WAVES

#### The Burgers equation

The simplest equation describing both nonlinear wave propagation and diffusion equation is the one-dimensional Burgers' equation,

$$c_t + cc_x = \nu c_{xx} \quad (2.1)$$

As we've seen, this follows from the continuity equation  $\rho_t + j_x$  when  $j = J(\rho) - \nu \rho_x$ , with

$$c = J'(\rho) \text{ and } c''(\rho) = 0.$$

We have already obtained, a solution to Burgers equation in the form of a propagating front. However, we can do much better than this; we can find all the solutions of the one-dimensional Burgers equation. The trick is to employ a nonlinear transformation of the field  $c(x,t)$ , known as the Cole-Hopf transformation, which linearizes the PDE. Once again, we follow the exceptionally clear discussion in the book by Whitham (ch. 4).

The Cole-Hopf transformation is defined as follows:

$$c \equiv -2\nu \frac{\varphi_x}{\varphi} = \frac{\partial}{\partial x} (-2\nu \ln \varphi) \quad (2.2)$$

Plugging into Burgers' equation, one finds that  $\varphi(x,t)$  satisfies the linear diffusion equation,

$$\varphi_t = \nu \varphi_{xx} \quad (2.3)$$

Isn't that just about the coolest thing you've ever heard?

Suppose the initial conditions on  $\varphi(x,t)$  are

$$\varphi(x,0) = \Phi(x). \quad (2.4)$$

We can then solve the diffusion equation 2.3 by Laplace transform. The result is

$$\varphi(x, t) = \frac{1}{\sqrt{4\pi vt}} \int_{-\infty}^{\infty} dx' e^{-(x-x')^2/4vt} \Phi(x') \quad (2.5)$$

Thus, if  $c(x, t=0) = g(x)$ , then the solution for subsequent times is

$$c(x, t) = \frac{\int_{-\infty}^{\infty} dx' (x-x') e^{-H(x, x', t)/2v}}{t \int_{-\infty}^{\infty} dx' e^{-H(x, x', t)/2v}}, \quad (2.6)$$

where

$$H(x, x', t) = \int_0^{x'} dx'' g(x'') + \frac{(x-x')^2}{2t}, \quad (2.7)$$

**The limit  $v \rightarrow 0$**

In the limit  $v \rightarrow 0$ , the integrals in the numerator and denominator of eqn. 2.6 may be computed via the method of steepest descents. This means that extremize  $H(x, x', t)$  with respect to  $x'$ , which entails solving

$$\frac{\partial H}{\partial x'} = g(x') - \frac{x-x'}{t} \quad (2.8)$$

Let  $\zeta = \zeta(x, t)$  be a solution to this equation for  $x'$ , so that

$$x = \zeta + g(\zeta)t \quad (2.9)$$

We now expand about  $x' = \zeta$ , writing  $x' = \zeta + s$ , in which case

$$H(x') = H(\zeta) + \frac{1}{2} H''(\zeta) s^2 + O(s^3) \quad (2.10)$$

where the  $x$  and  $t$  dependence is here implicit. If  $F(x')$  is an arbitrary function which is slowly varying on distance scales on the order of  $v^{1/2}$ , then we have

$$\int_{-\infty}^{\infty} dx' F(x') e^{-H(x')/2v} \approx \sqrt{\frac{4\pi v}{H''(\zeta)}} e^{-H(\zeta)/2v} F(\zeta) \quad (2.11)$$

Applying this result to eqn. 2.6, we find

$$c \approx \frac{x-\zeta}{t} \quad (2.12)$$

which is to say

$$c = g(\zeta) \quad (2.13)$$

$$x = \zeta + g(\zeta)t \quad (2.14)$$

This is precisely what we found for the characteristics of  $c_t + cc_x = 0$ .

What about multivaluedness? This is obviated by the presence of an additional saddle point solution. I.e. beyond some critical time, we have a discontinuous change of saddles as a function of  $x$ :

$$x = \zeta_{\pm} + g(\zeta_{\pm})t \rightarrow \zeta_{\pm} = \zeta_{\pm}(x, t) \quad (2.15)$$

Then

$$c \sim \frac{1}{t} \cdot \frac{\frac{x - \zeta_-}{\sqrt{H''(\zeta_-)}} e^{-H(\zeta_-)/2\nu} + \frac{x - \zeta_+}{\sqrt{H''(\zeta_+)}} e^{-H(\zeta_+)/2\nu}}{\frac{1}{\sqrt{H''(\zeta_-)}} e^{-H(\zeta_-)/2\nu} + \frac{1}{\sqrt{H''(\zeta_+)}} e^{-H(\zeta_+)/2\nu}} \quad (2.16)$$

Thus,

$$H(\zeta_+) > H(\zeta_-) \Rightarrow c \approx \frac{x - \zeta_-}{t} \quad (2.17)$$

$$H(\zeta_+) < H(\zeta_-) \Rightarrow c \approx \frac{x - \zeta_+}{t} \quad (2.18)$$

At the shock, these solutions are degenerate:

$$H(\zeta_+) = H(\zeta_-) \Rightarrow \frac{1}{2}(\zeta_+ - \zeta_-)(g(\zeta_+) + g(\zeta_-)) = \int_{\zeta_-}^{\zeta_+} d\zeta g(\zeta), \quad (2.19)$$

which is again exactly as before. We stress that for  $\nu$  small but finite the shock fronts are smoothed out on a distance scale proportional to  $\nu$ .

What does it mean for  $\nu$  to be small? The dimensions of  $\nu$  are  $[\nu] = L^2/T$ , so we must find some other quantity in the problem with these dimensions. The desired quantity is the area,

$$A = \int_{-\infty}^{\infty} dx [g(x) - c_0], \quad (2.20)$$

where  $c_0 = c(x = \pm\infty)$ . We can now define the dimensionless ratio,

$$R \equiv \frac{A}{2\nu}, \quad (2.21)$$



which is analogous to the Reynolds number in viscous fluid flow.  $R$  is proportional to the ratio of the nonlinear term  $(c - c_0)c_x$  to the diffusion term  $\nu c_{xx}$ .

### Examples of Burger's equation

Whitham discusses three examples: diffusion of an initial step, a hump, and an N-wave. Here we simply reproduce the functional forms of these solutions. For details, see chapter 4 of Whitham's book.

For an initial step configuration,

$$c(x, t = 0) = \begin{cases} c_1, & x < 0 \\ c_2, & x > 0 \end{cases} \quad (2.22)$$

We are interested in the case  $c_1 > c_2$ . Using the Cole-Hopf transformation and applying the appropriate initial conditions to the resulting linear diffusion equation, one obtains the complete solution,

$$c(x, t) = c_2 + \frac{c_1 - c_2}{1 + h(x, t) \exp[(c_1 - c_2)(x - v_s t) / 2\nu]}, \quad (2.23)$$

where

$$v_s = \frac{1}{2}(c_1 + c_2) \quad (2.24)$$

and

$$h(x, t) = \frac{\operatorname{erfc}\left(-\frac{x - c_2 t}{\sqrt{4\nu t}}\right)}{\operatorname{erfc}\left(+\frac{x - c_1 t}{\sqrt{4\nu t}}\right)} \quad (2.25)$$

Recall that  $\operatorname{erfc}(z)$  is the complimentary error function:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du e^{-u^2} \quad (2.26)$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty du e^{-u^2} = 1 - \operatorname{erfc}(z). \quad (2.27)$$

Note the limiting values  $\operatorname{erfc}(-\infty)=2, \operatorname{erfc}(0)=1$  and  $\operatorname{erfc}(\infty)=0$ . If  $c_2 < x/t < c_1$ , then  $h(x,t) \rightarrow 1$  as  $t \rightarrow \infty$ , in which case the solution resembles a propagating front. It is convenient to dimensionalize  $(x,t) \rightarrow (y,\tau)$  by writing

$$x = \frac{vy}{\sqrt{c_1 c_2}}, t = \frac{vt}{c_1 c_2}, r \equiv \sqrt{\frac{c_1}{c_2}} \quad (2.28)$$

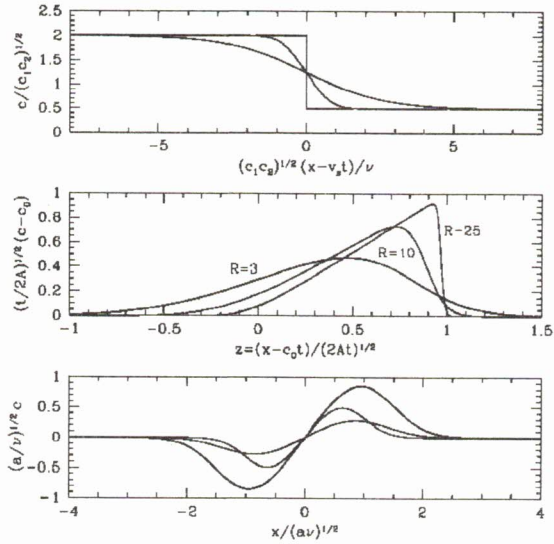


Fig. 2.1. Evolution of profiles of Burger's equation. Top: a step discontinuity evolving into a front at times  $\tau=0, \tau=\frac{1}{5}$  and  $\tau=5$ . Middle: narrow bump  $c_0 + A\delta(x)$  evolves into a triangular wave. Bottom: dissipation of an N-wave at times  $\tau = \frac{1}{4}, \tau = \frac{1}{2}, \tau = 1$ .

We then have

$$\frac{c(z,\tau)}{\sqrt{c_1 c_2}} = r^{-1} + \frac{2\alpha}{1 + h(z,\tau)\exp(\alpha z)}, \quad (2.29)$$

where

$$h(z, \tau) = \operatorname{erfc}\left(-\frac{z + \alpha \tau}{2\sqrt{\tau}}\right) / \operatorname{erfc}\left(\frac{z - \alpha \tau}{2\sqrt{\tau}}\right) \quad (2.30)$$

and

$$\alpha \equiv \frac{1}{2}(r - r^{-1}), z \equiv y - \frac{1}{2}(r + r^{-1})x \quad (2.31)$$

The second example involves the evolution of an infinitely thin hump, where

$$c(x, t = 0) = c_0 + A\delta(x) \quad (2.32)$$

The solution for subsequent times is

$$c(x, t) = c_0 + \sqrt{\frac{v}{\pi}} \cdot \frac{(e^{\pi} - 1) \exp\left(-\frac{x - c_0 t}{4vt}\right)}{1 + \frac{1}{2}(e^{\pi} - 1) \operatorname{erfc}\left(\frac{x - c_0 t}{4vt}\right)}, \quad (2.33)$$

where  $R = A/2v$ . Defining

$$z \equiv \frac{x - c_0 t}{\sqrt{2At}}, \quad (2.34)$$

we have the solution

$$c = c_0 + \left(\frac{2A}{t}\right)^{1/2} \frac{1}{\sqrt{4\pi R}} \cdot \frac{(e^{\pi} - 1)e^{-Rz^2}}{1 + \frac{1}{2}(e^{\pi} - 1) \operatorname{erfc}(\sqrt{R}z)}. \quad (2.35)$$

Asymptotically, for  $t \rightarrow \infty$  with  $x/t$  fixed, we have

$$c(x, t) = \begin{cases} x/t, & 0 < x < \sqrt{2At} \\ 0 & \end{cases}, \quad (2.36)$$

This recapitulates the triangular wave solution with the two counterpropagating shock fronts and dissipating shock strengths.

Finally, there is the N-wave. If we take the following solution to the linear diffusion equation,

$$\varphi(x, t) = 1 + \sqrt{\frac{a}{t}} e^{-x^2/4vt}, \quad (2.37)$$

then we obtain

$$c(x,t) = \frac{x}{t} \cdot \frac{e^{-x^2/4vt}}{\sqrt{\frac{a}{t} + e^{-x^2/4vt}}} \quad (2.38)$$

In terms of dimensionless variables  $(y, \tau)$  where

$$x = \sqrt{av}y, t = \alpha\tau \quad (2.39)$$

we have

$$c = \sqrt{\frac{v}{a}} \frac{y}{\tau} \cdot \frac{e^{-y^2/4\tau}}{\sqrt{\tau + e^{-y^2/4\tau}}} \quad (2.40)$$

The evolving profiles for these three cases are plotted in fig. 2.1.

### The Korteweg-deVries Equation

Let  $h_0$  denote the resting depth of water in a one-dimensional channel, and  $y(x, t)$  the vertical displacement of the water's surface. Let  $L$  be a typical horizontal scale of the wave. When, and  $|y| \ll h_0, h_0^2 \ll L^2$ , and,  $v \approx 0$  the evolution of an x-directed wave is described by the KdV equation,

$$y_t + c_0 y_x + \frac{3c_0}{2h_0} y y_x + \frac{1}{6} c_0 h_0^2 y_{xxx} = 0, \quad (2.41)$$

where  $c_0 = \sqrt{gh_0}$ . For small amplitude disturbances, only the first two terms are conse-

quential, and we have

$$y_t + c_0 y_x \approx 0, \quad (2.42)$$

the solution to which is

$$y(x, t) = f(x - c_0 t), \quad (2.43)$$

where  $f(\xi)$  is an arbitrary shape; the disturbance propagates with velocity. When the dispersion

and nonlinearity are included, only a particular pulse shape can propagate in an undistorted manner; this is the soliton.

It is convenient to shift to a moving frame of reference:

$$\tilde{x} = x - c_0 t, \quad \tilde{t} = t \quad (2.44)$$

hence

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} = -c_0 \frac{\partial}{\partial \bar{x}} \quad (2.45)$$

Thus,

$$y_{\bar{t}} + c_0 y_{\bar{x}} + \frac{3c_0}{2h} y y_{\bar{x}} + \frac{1}{6} c_0 h^2 y_{\bar{x}\bar{x}\bar{x}} = 0 \quad (2.46)$$

Finally, rescaling position, time, and displacement, we arrive at the KdV equation ,

$$u_t + 6uu_x + u_{xxx} = 0 \quad (2.47)$$

which is a convenient form.

### KdV solitons

We seek a solution to the KdV equation of the form  $u(x, t) = u(x - Vt)$ . Then with,  $\xi = x - Vt$  we have  $\partial_x = \partial_\xi$  and  $\partial_t = -V\partial_\xi$  when acting on  $u(x, t) = u(\xi)$ . Thus, we have

$$-Vu' + 6uu' + u''' = 0 \quad (2.48)$$

Integrating once, we have

$$-Vu + 3u^2 + u'' = A \quad (2.49)$$

where A is a constant. We can integrate once more, obtaining

$$-\frac{1}{2}Vu^2 + u^3 + \frac{1}{2}(u')^2 = Au + B \quad (2.50)$$

where now both A and B are constants. We assume that u and all its derivatives vanish in

the limit  $\xi \rightarrow \pm\infty$ , which entails  $A = B = 0$ . Thus,

$$\frac{du}{d\xi} = \pm u\sqrt{V - 2u} \quad (2.51)$$

With the substitution

$$u = \frac{1}{2}V \operatorname{sech}^2 \Theta \quad (2.52)$$

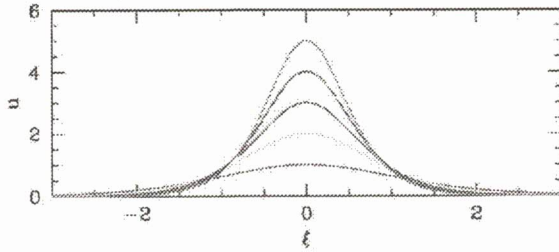


Fig. 2.2: Soliton solution to the KdV equation, with five eventually spaced  $V$  values ranging from  $V=2$  to  $V=10$ . The greater the speed, the narrower the shape

we find  $d\Theta = \mp \frac{1}{2} \sqrt{V} d\xi$ , hence we have the solution

$$u(x,t) = \frac{1}{2} V \operatorname{sech}^2 \left( \frac{\sqrt{V}}{2} (x - Vt - \xi_0) \right) \quad (2.53)$$

Note that the maximum amplitude of the soliton is  $u_{\max} = \frac{1}{2} V$ , which is proportional to its velocity  $V$ . The KdV equation imposes no limitations on  $V$  other than  $V \geq 0$ .

### Periodic solutions : soliton trains

If we relax the condition  $A = B = 0$ , new solutions to the KdV equation arise. Define the cubic

$$P(u) = 2u^3 - Vu^2 - 2Au - 2B \equiv 2(u - u_1)(u - u_2)(u - u_3), \quad (2.54)$$

where  $u_i = u_i(A, B, V)$ . We presume that  $A$ ,  $B$ , and  $V$  are such that all three roots  $u_{1,2,3}$  are real and nondegenerate. Without further loss of generality, we may then assume  $u_1 < u_2 < u_3$ . Then

$$\frac{du}{d\xi} = \pm \sqrt{-P(u)} \quad (2.55)$$

Since  $P(u) < 0$  for  $u_2 < u < u_3$ , we conclude  $u(\xi)$  must lie within this range.

Therefore, we

Have

$$\xi - \xi_0 = \pm \int_{u_2}^u \frac{ds}{\sqrt{-P(s)}} = \pm \left( \frac{2}{u_3 - u_1} \right)^{1/2} \int_0^\phi \frac{d\Theta}{\sqrt{1 - k^2 \sin^2 \Theta}} \quad (2.56)$$

where

$$u \equiv u_3 - (u_3 - u_2) \sin^2 \phi \quad (2.57)$$

$$k^2 \equiv \frac{u_3 - u_2}{u_3 - u_1} \quad (2.58)$$

The solution for  $u(\xi)$  is then

$$u(\xi) = u_3 - (u_3 - u_2) \operatorname{sn}^2(\zeta, k) \quad (2.59)$$

where

$$\zeta = \sqrt{\frac{u_3 - u_1}{2}} (\xi - \xi_0) \quad (2.60)$$

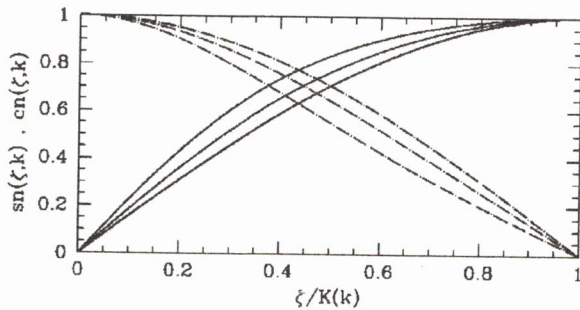


Fig 2.3: The Jacobi elliptic functions  $\operatorname{sn}(\zeta, k)$  (solid) and  $\operatorname{cn}(\zeta, k)$

(dot-dash) versus  $\zeta/K(k)$ , for  $k=0$  (blue),  $k = \frac{1}{\sqrt{2}}$ , and  $k=0,9$

and  $\operatorname{sn}(\zeta, k)$  is the Jacobi elliptic function.

### Interlude: primer on elliptic functions

We assume  $0 \leq k^2 \leq 1$  and we define

$$\zeta(\phi, k) = \int_0^\phi \frac{d\Theta}{\sqrt{1-k^2 \sin^2 \Theta}} \quad (2.61)$$

The sn and cn functions are defined by the relations

$$sn(\zeta, k) = \sin \phi \quad (2.62)$$

$$cn(\zeta, k) = \cos \phi \quad (2.63)$$

Note that  $sn^2(\zeta, k) + cn^2(\zeta, k) = 1$ . One also defines the function  $dn(\zeta, k)$  from the relation

$$dn^2(\zeta, k) + k^2 sn^2(\zeta, k) = 1 \quad (2.64)$$

When  $\phi$  advances by one period, we have  $\Delta\phi = 2\pi$ , and therefore  $\Delta\zeta = Z$ ,

where

$$Z = \int_0^{2\pi} \frac{d\Theta}{\sqrt{1-k^2 \sin^2 \Theta}} = 4K(k) \quad (2.65)$$

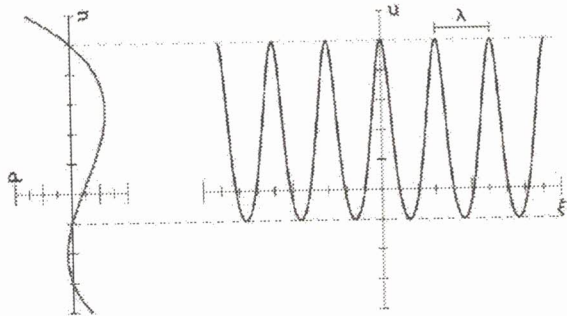


Fig. 2.4 The cubic function  $P(u)$  (left), and the solution lattice (right) for the case  $u_1 = -1.5$ ,  $u_2 = -0.5$ ,  $u_3 = -2.5$ .

where  $K(k)$  is the complete elliptic integral of the first kind. Thus,  $sn(\zeta + Z, k) = sn(\zeta, k)$  and similarly for the cn function. In fig. 2.3, we sketch the behavior of the elliptic functions over one quarter of a period. Note that for  $k = 0$  we have  $sn(\zeta, 0) = \sin \zeta$  and  $cn(\zeta, 0) = \cos \zeta$



### The soliton lattice

Getting back to our solution in eqn.2.59, we see that the solution describes a soliton lattice with a wavelength

$$\lambda = \frac{\sqrt{8K(k)}}{\sqrt{u_3 - u_1}} \quad (2.66)$$

Note that our definition of  $P(u)$  entails

$$V = 2(u_1 + u_2 + u_3) \quad (2.67)$$

There is a simple mechanical analogy which merits illumination. Suppose we define

$$W(u) \equiv u^3 - \frac{1}{2}Vu^2 - Au \quad (2.68)$$

and furthermore  $E \equiv B$ . Then

$$\frac{1}{2} \left( \frac{du}{d\xi} \right)^2 + W(u) = E \quad (2.69)$$

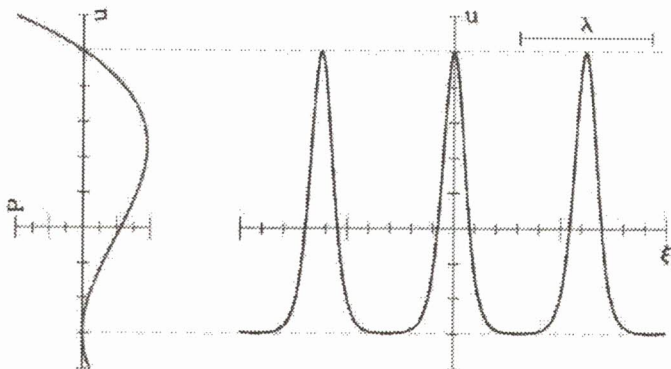


Fig. 2.5 The cubic function  $P(u)$  (left), and the solution lattice (right) for the case  $u_1=-1.5$ ,  $u_2=-1.49$ ,  $u_3=-2.50$

which takes the form of a one-dimensional Newtonian mechanical system, if we replace  $\xi \rightarrow t$  and interpret  $u_t$  as a velocity. The potential is  $W(u)$  and the total energy is  $E$ . In terms of the polynomial  $P(u)$ , we have  $P = 2(W - E)$ . Accordingly, the 'motion'  $u(\xi)$  is flattest for the lowest values of  $u$ , near  $u = u_2$ , which is closest to the local maximum of  $W(u)$ .

Note that specifying  $u_{\min} = u_2$ ,  $u_{\max} = u_3$ , and the velocity  $V$  specifies all the parameters. Thus, we have a three parameter family of soliton lattice solutions.

### N-soliton solutions to KdV

In 1971, Ryogo Hirota showed that exact N-soliton solutions to the KdV equation exist. Here we discuss the Hirota solution, following the discussion in the book by Whitham. The KdV equation may be written as

$$u_t + \{3u^2 + u_{xx}\}_x = 0 \quad (2.70)$$

which is in the form of the one-dimensional continuity equation  $u_t + j_x = 0$ , where the current is  $j = 3u^2 + u_{xx}$ . Let us define  $u = p_x$ . Then our continuity equation reads  $p_x + j_x = 0$ , which can be integrated to yield  $p_t + j = C$ , where  $C$  is a constant. Demanding that  $u$  and its derivatives vanish at spatial infinity requires  $C = 0$ . Hence, we have

$$p_t + 3p_x^2 + p_{xxx} = 0 \quad (2.71)$$

Now consider the nonlinear transformation

$$p = 2(\ln F)_x = \frac{2F_x}{F} \quad (2.72)$$

We then have

$$p_t = \frac{2F_{xt}}{F} - \frac{2F_x F_t}{F^2} \quad (2.73)$$

$$p_x = \frac{2F_{xx}}{F} - \frac{2F_x^2}{F^2} \quad (2.74)$$

and

$$p_{xx} = \frac{2F_{xxx}}{F} - \frac{6F_x F_{xx}}{F^2} + \frac{4F_x^3}{F^3} \quad (2.75)$$

$$p_{xxx} = \frac{2F_{xxxx}}{F} - \frac{8F_x F_{xxx}}{F^2} - \frac{6F_x^2 x^2}{F^2} + \frac{24F_x^2 F_{xx}}{F^3} - \frac{12F_x^4}{F^4} \quad (2.76)$$

When we add up the combination  $p_t + 3p_x^2 + p_{xxx} = 0$ , we find, remarkably, that the terms with  $F^3$  and  $F^4$  in the denominator cancel. We are then left with

$$F(F_t + F_{xxx})_x - F_x(F_t + F_{xxx}) + 3(F_{xx}^2 - F_x F_{xxx}) = 0 \quad (2.77)$$

This equation has the two-parameter family of solutions

$$F(x, t) = 1 + e^{\phi(x, t)} \quad (2.78)$$

where

$$\phi(x, t) = \alpha(x - b - \alpha^2 t), \quad (2.79)$$

with  $\alpha$  and  $b$  constants. Note that these solutions are all annihilated by the operator  $\partial_t + \partial_x^3$ , and also by the last term in eqn. 2.77 because of the homogeneity of the derivatives. Converting back to our original field variable  $u(x, t)$ , we have that these solutions are single solitons:

$$u = p_x = \frac{2(F F_{xx} - F_x^2)}{F^2} = \frac{\alpha^2 f}{(1+f)^2} = \frac{1}{2} \alpha^2 \operatorname{sech}^2\left(\frac{1}{2}\phi\right) \quad (2.80)$$

The velocity for these solutions is  $V = \alpha^2$ .

If eqn. 2.77 were linear, our job would be done, and we could superpose solutions. We will meet up with such a felicitous situation when we discuss the Cole-Hopf transformation for the one-dimensional Burgers' equation. But for KdV the situation is significantly more difficult. We will write

$$F = 1 + F^{(1)} + F^{(2)} + \dots + F^{(N)}, \quad (2.81)$$

with

$$F^{(1)} = f_1 + f_2 + \dots + f_N, \quad (2.82)$$

where

$$f_j(x, t) = e^{\phi_j(x, t)} \quad (2.83)$$

$$\phi_j(x,t) = \alpha_j(x - \alpha_j^2 t - b_j) \quad (2.84)$$

We may then derive a hierarchy of equations, the first two levels of which are

$$(F_t^{(1)} + F_{xxx}^{(1)})_x = 0 \quad (2.85)$$

$$(F_t^{(2)} + F_{xxx}^{(2)})_x = -3(F_{xx}^{(1)}F_{xx}^{(1)} - F_x^{(1)}F_{xxx}^{(1)}) \quad (2.86)$$

Let's explore the case  $N = 2$ . The equation for  $F(2)$  becomes

$$(F_t^{(2)} + F_{xxx}^{(2)})_x = 3\alpha_1\alpha_2(\alpha_2 - \alpha_1)^2 f_1 f_2, \quad (2.87)$$

with solution

$$F^{(2)} = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \right)^2 f_1 f_2. \quad (2.88)$$

Remarkably, this completes the hierarchy for  $N = 2$ . Thus,

$$F = 1 + f_1 + f_2 + \left( \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \right)^2 f_1 f_2 = \det \begin{pmatrix} 1 + f_1 & \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1 + \alpha_2} f_2 \\ \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1 + \alpha_2} f_2 & 1 + f_2 \end{pmatrix}. \quad (2.89)$$

What Hirota showed, quite amazingly, is that this result generalizes to the  $N$ -soliton case,

$$F = \det Q \quad (2.90)$$

where  $Q$  is the symmetric matrix,

$$Q_{mn} = \delta_{mn} + \frac{2\sqrt{\alpha_m\alpha_n}}{\alpha_m + \alpha_n} f_m f_n \quad (2.91)$$

Thus,  $N$ -soliton solutions to the KdV equation may be written in the form

$$u(x,t) = 2 \frac{\partial^2}{\partial x^2} \ln \det Q(x,t). \quad (2.92)$$

Consider the case  $N = 2$ . Direct, if tedious, calculations lead to the expression

$$u = 2 \frac{\alpha_1^2 f_1 + \alpha_2^2 f_2 + 2(\alpha_1 - \alpha_2)^2 f_1 f_2 + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 (\alpha_1^2 f_1 f_2^2 + \alpha_2^2 f_1^2 f_2)}{\left[1 + f_1 + f_2 + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 f_1 f_2\right]^2} \quad (2.93)$$

Recall that

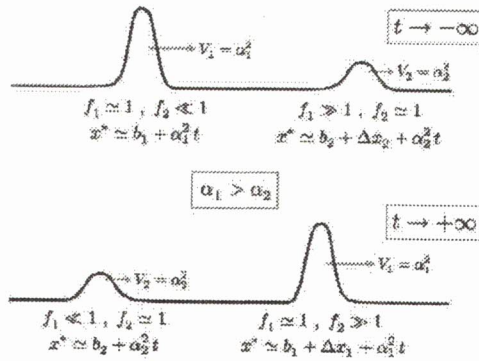


Fig. 2.6 Early and late time configuration of the two soliton solution of the KdV equation

$$f_j(x,t) = \exp[\alpha_j(x_j - \alpha_j^2 t - b_j)] \quad (2.94)$$

Let's consider  $(x, t)$  values for which  $f_1 \approx 1$  is neither large nor small, and investigate what happens in the limits  $f_2 \ll 1$  and  $f_2 \gg 1$ . In the former case, we find

$$u \approx \frac{2\alpha_1^2 f_1}{(1 + f_1)^2} \quad (f_2 \ll 1), \quad (2.95)$$

which is identical to the single soliton case of eqn. 10.43. In the opposite limit, we have

$$u \approx \frac{2\alpha_1^2 g_1}{(1 + g_1)^2} \quad (f_2 \gg 1), \quad (2.96)$$

where

$$g_1 = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \right)^2 f_1 \quad (2.97)$$

But multiplication of  $f_j$  by a constant  $C$  is equivalent to a translation:

$$Cf_j(x,t) = f_j(x + \alpha_j^{-1} \ln C, t) \equiv f_j(x - \Delta x_j, t) \quad (2.98)$$

Thus, depending on whether  $f_2$  is large or small, the solution either acquires or does not acquire a spatial shift  $\Delta x_1$ , where

$$\Delta x_j = \frac{2}{\alpha_j} \ln \left| \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \right|. \quad (2.99)$$

The function  $f(x,t) = \exp[\alpha(x - \alpha^2 t - b)]$  is monotonically increasing in  $x$  (assuming  $\alpha > 0$ ). Thus, if at fixed  $t$  the spatial coordinate  $x$  is such that  $f \ll 1$ , this means that the soliton lies to the right. Conversely, if  $f \gg 1$  the soliton lies to the left. Suppose  $\alpha_1 > \alpha_2$ , in which case soliton #1 is stronger (i.e. greater amplitude) and faster than soliton #2. The situation is as depicted in figs. 10.5 and 10.6. Starting at early times, the strong soliton lies to the left of the weak soliton. It moves faster, hence it eventually overtakes the weak soliton. As the strong soliton passes through the weak one, it is shifted forward, and the weak soliton is shifted backward. It hardly seems fair that the strong fast soliton gets pushed even further ahead at the expense of the weak slow one, but sometimes life is just like that.

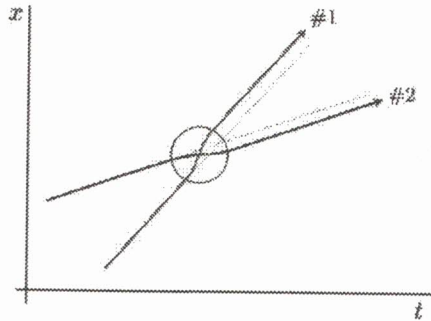
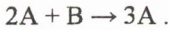


Fig 2.7. Spacetime diagram for the collision of two KdV solitons. The strong fast soliton (#1) is shifted forward and the weak slow one (#2) is shifted backward/ The lines indicate the centers of the two solitons. The shaded circle is the 'interaction region' where the solitons is not simply a sum of two single soliton wavefronts.

## 2.2 PATTERN FORMATION

### The Brusselator

Consider the so-called Brusselator model of Prigogine and Lefever (1968). The Brusselator is a model for two fictitious chemical reactions,



The species A is assumed to be supplied and removed from the system, in which case, after

adding diffusion, we have two coupled RDEs with

$$f(u, v) = a - (1 + b)u + u^2v \quad (2.100)$$

$$g(u, v) = bu - u^2v. \quad (2.101)$$

The fixed point  $f = g = 0$  occurs at  $(u^*, v^*) = (a, b/a)$ . Linearizing the local dynamics about the fixed point, we obtain

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix} \quad (2.102)$$

Thus,  $\Delta = a^2 > 0$ . The Hopf instability sets in when  $f_u + g_v = 0$ , i.e. when  $b = b_H$ , where

$$b_H = 1 + a^2. \quad (2.103)$$

For the Turing instability,

$$D_u g_v + D_v f_u = 2\sqrt{\Delta D_u D_v} \quad (2.104)$$

gives  $b = b_T$ , where

$$b_T = (1 + c)^2, \quad (2.105)$$

where we have defined

$$c \equiv a \sqrt{\frac{D_u}{D_v}} \quad (2.106)$$

Note that  $c < a$  for the Turing instability. These two curves intersect at

$$c^* = -1 + \sqrt{1 + a^2} \quad (2.107)$$



Note that

$$Q^2 = \frac{a}{\sqrt{D_u D_v}} \Rightarrow D_u Q^2 = c, D_v Q^2 = \frac{a^2}{c} \quad (2.108)$$

Suppose we are close to the Turing instability, and we write  $b = b_T + \epsilon$  with  $|\epsilon| \ll 1$ . We first expand the coupled RDEs about the fixed point, writing

$$u = u^* + \delta u \quad (2.109)$$

$$v = v^* + \delta v \quad (2.110)$$

with  $u^* = a$  and  $v^* = \frac{b}{a}$ . Written in terms of  $\delta u$  and  $\delta v$ , the coupled RDEs take the form

$$\begin{aligned} \delta u_t &= D_u \delta u_{xx} - \delta u + (b \delta u + a^2 \delta v) + \left( \frac{b}{a} (\delta u)^2 + 2a \delta u \delta v + (\delta v)^2 \delta v \right) \\ \delta v_t &= D_v \delta v_{xx} - (b \delta u + a^2 \delta v) - \left( \frac{b}{a} (\delta u)^2 + 2a \delta u \delta v + (\delta v)^2 \delta v \right) \end{aligned} \quad (2.111)$$

If we ignore the nonlinear terms, we obtain a linear equation which has a solution

$$\delta u(x, t) = U_{11} A(t) \cos(Qx) \quad (2.112)$$

$$\delta v(x, t) = V_{11} A(t) \cos(Qx), \quad (2.113)$$

where  $A(t) = A_0 \exp(\omega t)$  is an amplitude, and where the eigenvector  $(U_{11} \ V_{11})^t$  satisfies

$$\begin{pmatrix} b-c-1 & a^2 \\ -b & -a^2 - \frac{a^2}{c} \end{pmatrix} \begin{pmatrix} U_{11} \\ V_{11} \end{pmatrix} = \omega \begin{pmatrix} U_{11} \\ V_{11} \end{pmatrix} \quad (2.114)$$

If  $b > b_T$ , then there exists an eigenvalue  $\omega$  which is real and positive, in which case the amplitude  $A(t)$  grows exponentially.

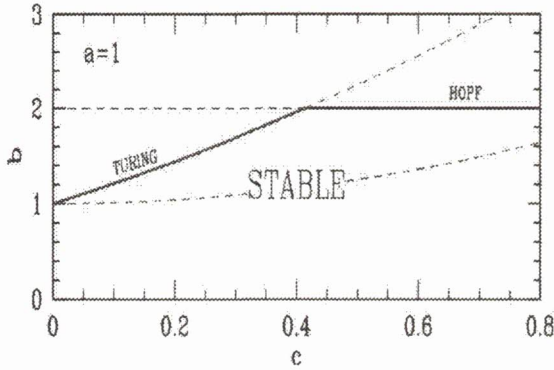


Figure 2.8: Instability lines for the Brusselator. The thick lines denote a Hopf instability and a Turing instability. The dashed light line is the locus of points for which  $Q^2 > 0$ .

The amplitude equation has a fixed point when  $At = 0$ , which says  $\eta = gA^2$ .

### The amplitude equation

The exponential growth of the amplitude  $A(t)$  is valid only insofar as the nonlinear terms in the dynamics are small. Our goal here will be to develop a nonlinear ODE governing the growth of  $A(t)$ , assuming  $|b - bT| \ll 1$ . We follow the treatment of Kessler and Levine.

We assume one Fourier mode will be excited, with  $q = \pm Q$ , along with its harmonics. We therefore write

$$\delta u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} U_{mn} A^m \cos(nQx) \quad (2.115)$$

$$\delta u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} V_{mn} A^m \cos(nQx) \quad (2.116)$$

We shall only need the first few terms:

$$\delta u = (U_{11}A + U_{31}A^3)\cos(Qx) + U_{20}A^2 + U_{22}A^2 \cos(2Qx) + \dots \quad (2.117)$$

$$\delta u = (V_{11}A + V_{31}A^3)\cos(Qx) + V_{20}A^2 + V_{22}A^2 \cos(2Qx) + \dots \quad (2.118)$$

Note that we assume  $U_{10} = V_{10} = 0$  because the leading behavior is in the  $U_{11}$  and  $V_{11}$  terms. It is through the quadratic nonlinearities that terms with  $n = 0$  are generated. We now undertake the tedious process of working out the RHS of eqns. 2.111 to order  $A^3$ . Throughout our derivation, we shall include only the  $n = 0$ ,  $n = 1$  and  $n = 2$  harmonics and drop all other terms. We will also assume  $b = b_T$  whenever it multiplies  $A^m$  with  $m > 1$ , since  $\epsilon = b - b_T$  is presumed small, and, as we shall see, the amplitude itself will be proportional to  $\sqrt{\epsilon}$ . Let's roll up our sleeves and get to work!

The first terms we need are all the ones linear in  $\delta u$  and  $\delta v$ . Thus, we need

$$D_u \delta u_{xx} - \delta u = -(1+c)(U_{11}A + U_{31}A^3)\cos(Qx) - U_{20}A^2 - (1+4c)U_{22}A^2 \cos(2Qx) \quad (2.119)$$

$$D_u \delta u_{xx} - \delta u = -\frac{a^2}{c}(V_{11}A + V_{31}A^3)\cos(Qx) - \frac{4a^2}{c}V_{22}A^2 \cos(2Qx) \quad (2.120)$$

as well as

$$b\delta u + a^2\delta v = \left\{ b(U_{11}A + U_{31}A^3) + a^2(V_{11}A + V_{31}A^3) \right\} \cos(Qx) + (bU_{20} + a^2V_{20})A^2 + (bU_{22} + a^2V_{22})A^2 \cos(2Qx) \quad (2.121)$$

Next, we need the nonlinear terms, starting with

$$(\delta u)^2 = \frac{1}{2}U_{11}^2A^2 + \frac{1}{2}U_{11}^2A^2 \cos(2Qx) + U_{11}(2U_{20} + U_{22})A^3 \cos(Qx) + \dots \quad (2.122)$$

where the remaining terms are of  $O(A^4)$  or are proportional to  $\cos(3Qx)$ .

We also require

$$2\delta u\delta v = U_{11}V_{11}A^2 + U_{11}V_{11}A^2 \cos(2Qx) + (2U_{11}V_{20} + 2V_{11}U_{20} + U_{11}V_{22} + V_{11}U_{22})A^3 \cos(Qx) + \dots \quad (2.123)$$

Finally, we need

$$(\delta u)^2 \delta v = \frac{3}{4}U_{11}^2V_{11} \cos(Qx) \dots \quad (2.124)$$

On the left hand side of eqns.2.111, we have the time derivative terms. Again, as we shall see, the amplitude  $A$  will be proportional to  $\sqrt{\epsilon}$ , where  $\epsilon = b - b_T$  is presumed small. Its time derivative  $A_t$  will be proportional to  $\epsilon^{3/2}$ .

Therefore, terms such as  $(A^2)_t = 2AA_t$  will be negligible and we shall drop them from the outset. Thus,

$$\delta u_t = U_{11} A_t \cos(Qx) + \dots \quad (2.125)$$

$$\delta v_t = V_{11} A_t \cos(Qx) + \dots \quad (2.126)$$

The linear terms in  $A$  comprise a zero mode for the system with  $b = b_T$ .

Thus,

$$\begin{pmatrix} b_t - c - 1 & a^2 \\ -b_t & -a^2 - \frac{a^2}{c} \end{pmatrix} \begin{pmatrix} U_{11} \\ V_{11} \end{pmatrix} = 0 \quad (2.127)$$

We may, without loss of generality, set  $U_{11} \equiv 1$ . We then have

$$U_{11} = 1, \quad V_{11} = -\frac{c(1+c)}{a^2} \quad (2.128)$$

We now set to zero the coefficients of  $\cos(nQx)$  for  $n = 0, 1$ , and  $2$  in each of eqns 2.111. Setting the  $n = 0$  terms on the RHS of these equations to zero, we obtain

$$U_{20} = 0 \quad (2.129)$$

$$\frac{b}{2a} U_{11}^2 + a U_{11} V_{11} + a^2 V_{20} = 0 \quad (2.130)$$

Doing the same for the  $n = 2$  terms, we get

$$\frac{b}{2a} U_{11}^2 + a U_{11} V_{11} + b U_{22} + a^2 (1 + 4c^{-1}) V_{22} = 0 \quad (2.131)$$

$$(1 + 4c) U_{22} + \frac{4a^2}{c} V_{22} = 0 \quad (2.132)$$

Solving, we obtain

$$\begin{aligned} U_{20} &= 0 & U_{22} &= \frac{2(1-c^2)}{9ac} \\ V_{20} &= -\frac{1-c^2}{2a^3} & V_{22} &= -\frac{(1-c^2)(1+4c)}{18a^3} \end{aligned} \quad (2.133)$$

Finally, we need the  $n = 1$  terms. There are three contributions. One comes from the linear terms, restoring the small differences proportional to  $\epsilon = b - b_T$ . These terms contribute a coefficient for  $\cos(Qx)$  of  $\phi A$  in the RHS of eqn.

9.34 and  $-\epsilon A$  on the RHS of eqn. 9.35. A second contribution comes from the nonlinear terms. We invoke eqns. 9.46 and 9.47, multiplying the former by  $\frac{b}{a}$  and the latter by  $a$ . The term we seek is proportional to  $A^3 \cos(Qx)$ , with a coefficient

$$\frac{b}{a}U_{22} + a(2V_{20} + V_{22} + V_{11}U_{22}) + \frac{3}{4}V_{11} = \frac{(1+c)(2+c)(8c^2 - 21c + 4)}{36a^2c} \quad (2.134)$$

Note that  $\lambda > 0$  for  $c \in [c_-, c_+]$ , where  $c_{\pm} = \frac{1}{16} (21 \pm \sqrt{313})$ . Numerically,  $c_- \approx 0.20676$  and  $c_+ \approx 2.4182$ . Finally, we have the  $U_{31}$  and  $V_{31}$  terms themselves. Thus, dividing out the common  $\cos(Qx)$  factor on both sides of both equations, we have

$$A_t = \epsilon A + [c(1+c)U_{31} + a^2V_{31} - \lambda]A_3 \quad (2.135)$$

$$-\frac{c(1+c)}{a^2}A_t = -\epsilon A - \left[ (1+c)^2U_{31} + \frac{a^2}{c}(1+c)V_{31} + \lambda \right]A_3 \quad (2.136)$$

We can rewrite these equations as a linear system for the coefficients  $U_{31}$  and  $V_{31}$ , viz.

$$\overbrace{\begin{pmatrix} c(1+c) & a^2 \\ -(1+c)^2 & -a^2 - \frac{a^2}{c} \end{pmatrix}}^M \begin{pmatrix} U_{31} \\ V_{31} \end{pmatrix} = \begin{pmatrix} A_t - \epsilon A + \lambda A^3 \\ -a^{-2}c(1+c)A_t + \epsilon A - \lambda A^3 \end{pmatrix} \quad (2.137)$$

In order to be able to satisfy the above equation, the RHS must be orthogonal to the left eigenvector of the matrix  $M$  corresponding to the zero eigenvalue. This is called the solvability condition. It is easy to see that this zero left eigenvector is proportional to

$$\varphi = (1 + c c) \quad (2.138)$$

Thus, we demand

$$(1+c)(A_t - \epsilon A + \lambda A^3) - c(a^{-2}c(1+c)A_t - \epsilon A + \lambda A^3) = 0 \quad (2.139)$$

This, at long last, yields our amplitude equation:

$$A_t = \eta A - g A^3, \quad (2.140)$$

Where

$$\eta = \frac{\alpha^2(b-b_1)}{(1+c)(a^2-c_2)} / , \quad g = -\frac{\alpha^2(2+c)(8c^2-21c+4)}{36\alpha^2c(a^2-c^2)} \quad (2.141)$$

Since  $c < a$ , we have that  $\eta$  is positive for  $b > bT$  and negative for  $b < bT$ . Furthermore  $g$  is positive for  $c \in [c_-, c_+]$  and negative outside this region. Thus,  $A$  has a fixed point

$A^* = \sqrt{\eta/g}$  (in addition to the one at  $A = 0$ ) if both  $\eta$  and  $g$  are positive, or if both  $\eta$  and  $g$  are negative. In the former case,  $A = 0$  is unstable and  $A = A^*$  is stable. In the latter case,  $A = 0$  is stable and  $A = A^*$  is unstable.

### Rayleigh-Benard Instability

Consider a layer of fluid between two horizontal plates, as depicted in fig. 2.9. The top plate is held at temperature  $T_1$  and the bottom plate at temperature  $T_2$ , with  $\Delta T = T_2 - T_1 > 0$ . As the fluid near the bottom plate is heated, it expands, and an upward buoyancy force per unit volume  $f_{\text{buoy}} = \rho g \alpha \Delta T$  results, where  $\alpha = \frac{1}{V} \frac{\partial V}{\partial T}$  is the thermal expansion coefficient and  $\rho$  is the fluid density. This buoyancy force is a destabilizing effect, and is opposed by a stabilizing dissipative force per unit volume  $f_{\text{diss}} = \nu \kappa \rho / d^3$ , where  $\nu$  is the kinematic viscosity,  $\kappa$  the thermal diffusivity, and  $d$  the distance between the plates. The dimensionless ratio of these two force densities is known as the Rayleigh number,

$$\mathfrak{R} = \frac{f_{\text{buoy}}}{f_{\text{diss}}} = \frac{gd^3 \alpha \Delta T}{\nu \kappa} \quad (2.142)$$

When  $\mathfrak{R} > \mathfrak{R}_c \approx 1708$ , the destabilizing effects are sufficient to destroy the homogeneous state. Due to mass conservation, the entire fluid cannot rise uniformly, hence the instability

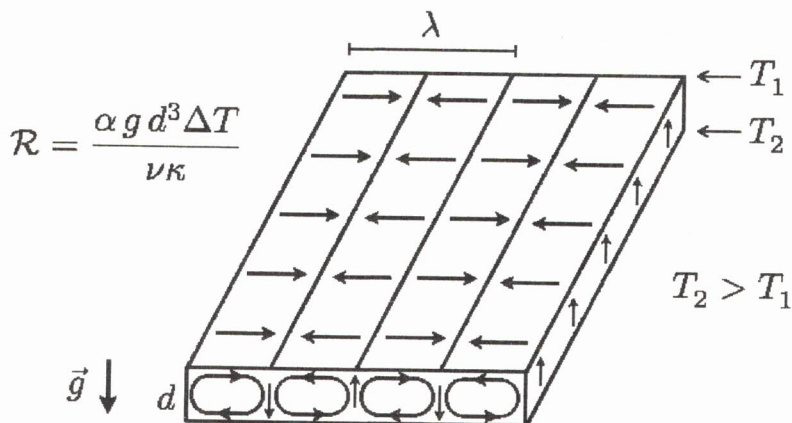


Figure 2.9: B'énard convection cells in a fluid heated from below.

occurs at a finite wavelength  $\lambda$ , owing to the formation of convective rolls known as B'énard cells (see fig.2.9).

Swift and Hohenberg (1977) showed that the dynamics for this problem reduces to the following equation for the real field  $\sigma(\mathbf{r}, t)^2$ :

$$\frac{\partial \sigma}{\partial t} = \left[ \varepsilon - (Q^2 + \nabla^2)^2 \right] \sigma - \sigma^3 \quad (2.143)$$

Here,

$$\varepsilon \propto \mathcal{R} - \mathcal{R}_c \quad (2.144)$$

measures the distance from the instability, and  $\nabla = \hat{x} \partial_x + \hat{y} \partial_y$  is the in-plane gradient.

Distances are measured in units of  $d$ , and the maximally unstable wavevector is  $Q \approx 3.12$ .

We assume a plane wave disturbance, and we first separate out the oscillating features of  $\sigma(\mathbf{r}, t)$  so that we may talk of a slowly varying amplitude function  $A(\mathbf{r}, t)$ :

$$\sigma^3 = A^3 e^{3iQx} + 3|A|^2 A e^{iQx} + 3|A|^2 A^* e^{-iQx} + A^{*3} e^{-3iQx} \quad (2.145)$$

$$(Q^2 + \nabla^2) A e^{iQx} = e^{iQx} (2iQ \partial_x + \partial_x^2 + \partial_y^2) A \quad (2.146)$$

Matching coefficients of  $e^{iQx}$ , we find

$$\partial_t A = \left\{ \epsilon - (2iQ\partial_x + \partial_x^2 + \partial_y^2)^2 \right\} A - 3|A|^2 A \quad (2.147)$$

If we assume that the solution for A is such that  $\partial_x < \epsilon^{1/2}$  and  $\partial_y < \epsilon^{1/4}$  when acting on A, then the  $\partial_x^2$  term is subleading relative to  $Q\partial_x$  and  $\partial_y^2$ , and we may drop it for  $|\epsilon| > 1$  and write

$$\partial_t A = \left\{ \epsilon - (2iQ\partial_x + \partial_y^2)^2 \right\} A - 3|A|^2 A \quad (2.148)$$

For  $\epsilon > 0$  there is a family of stationary solutions of the form

$$A(x, y) = A_q e^{iqx} e^{i\delta} \quad (2.149)$$

Where

$$A_q = \frac{1}{\sqrt{3}} \left( \epsilon - (2Qq_x + q_y^2)^2 \right)^{1/2} \quad (2.150)$$

The instability first occurs at  $q = 0$ , at  $\epsilon = 0$ . As we shall see, the nonlinearity severely limits what multimode structures may form.

The derivation of the Swift-Hohenberg equation utilizes something called the Boussinesq approximation, in which the density variation of the fluid enters only in the buoyancy equation. For non-Boussinesq fluids, the symmetry  $\sigma \rightarrow -\sigma$  is broken, and one has

$$\partial_t \sigma = \left[ \epsilon - (Q^2 + \nabla^2)^2 \right] \sigma + \nu \sigma^2 - \sigma^3 \quad (2.151)$$

for which the bifurcation is subcritical. This is called the Haken model .



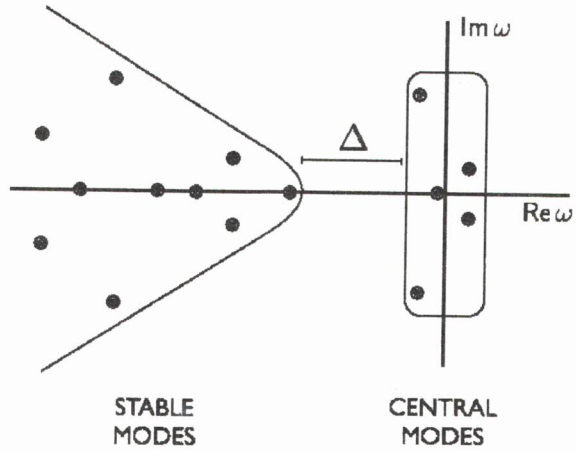


Figure2.10: Sketch showing separation of frequency scales owing to gap  $\Delta$ . The fast, stable modes are to the left, and the slow central modes all have  $\text{Re } \omega a \approx 0$ .

## 2.3 NONLINEAR STOCHASTIC PROCESSES

### **McKean and a New Class of Stochastic Processes.**

In two seminal papers, McKean introduced a new type of stochastic processes. The stochastic processes studied by McKean satisfy drift-diffusion equations for probability densities that are nonlinear with respect to their probability densities. While nonlinear drift-diffusion equations have frequently been used to describe how concentration fields evolve in time, the models considered by McKean describe stochastic processes. A stochastic process not only describes how expectation values (such as the mean value) evolve in time, but also, provides a complete description of all possible correlations between two or more than two time points. Consequently, the study of stochastic processes goes beyond the study of concentration fields or the evolution of single-time point expectation values such as the mean. The observation by McKean that nonlinear drift-diffusion equations can describe stochastic processes opened a new avenue of research.

### **Astrophysics: Stellar Cut-Off Mass Distributions Defined by Strongly Nonlinear Stochastic Models.**

Astrophysical mass distributions may exhibit relative sharp boundaries. More precisely, particle distributions in the stellar space may be spatially bounded due to gravity. The gravitational forces are produced by the mass distributions themselves. That is, the dynamics of an individual particle of a mass distribution is affected by the particle distribution as a whole. In turn, the distribution is composed of its individual particles. In view of this circular causality, it does not come as a surprise that strongly nonlinear stochastic models have been investigated that describe cut-off distributions of stellar particles. The models typically assume the form of the strongly nonlinear

Fokker-Planck equation. Two more comments may be appropriate. In addition to the aforementioned studies on strongly nonlinear processes describing astrophysical cut-off mass distribution, strongly nonlinear stochastic processes satisfying the Landau form have also been used for studying astrophysical problems.

### Material Physics of Porous Media.

Gas particles and fluids diffusing through porous media satisfy a nonlinear diffusion equation, frequently called the porous media equation. In one spatial dimension, the porous medium equation for the particle mass density  $\rho(x,t)$  reads

$$\frac{\partial}{\partial t} \rho(x,t) = A \frac{\partial^2}{\partial x^2} [\rho(x,t)]^q, \tag{2.152}$$

with  $A > 0$ . Dividing the mass density  $\rho(x,t)$  by the total mass  $M$ , (2.152) becomes an evolution equation for a probability density  $P(x,t)$  normalized to unity

$$\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} [P(x,t)]^q, \tag{2.153}$$

with  $D = A \cdot M^{(q-1)}$  and assumes the form of the nonlinear Fokker-Planck equation. The connection between the porous medium equation and nonlinear master equations has been addressed in Section 5.1 (see the aforementioned). Equations (2.152) and (2.153) exhibit exact time-dependent solutions, frequently called Barenblatt-Pattle solutions. The parameter  $q$  captures the nonlinearity of the equation. For  $q = 1$  (linear case) the porous medium equation reduces to the ordinary diffusion equations and the variance of the diffusion process (or the second moment of  $P(x,t)$  assuming a zero first moment) increases linearly with time  $t$ . In contrast, for  $q > 1/3$  and  $q \neq 1$  (nonlinear case) the variance increases as a nonlinear function of  $t$  like  $t^r$  with  $r = 2/(1+q)$  as can be shown by various methods. Interpreting (2.153) in terms of a strongly nonlinear Fokker-Planck

equation, trajectories of realizations can be computed from the corresponding strongly nonlinear Langevin equation. For a generalized version of (2.153) involving a drift force such a Langevin equation approach has been exploited to derive analytical expressions of certain autocorrelation functions. The porous medium equation in its form as a nonlinear Fokker-Planck equation (2.153) plays an important role in the theory development of non extensive thermostatics. As it turns out, a particular generalization of (2.153), the so-called Plastino-Plastino model, exhibits stationary solutions that maximize the non extensive entropy proposed by Tsallis.

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Одеський державний екологічний університет  
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