THE MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

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Methodical instructions for self-sufficient work of PhD students, tests performance and distance learning in the discipline «Fractal geometry and a chaos theory», Part 4

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Methodical instructions for self-studying work of the second-year PhD students and tests performance in the discipline «Fractal Geometry and Theory of a Chaos», Part 4

(Training specialty – 113 “Applied Mathematics” and others)

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PREFACE

The discipline "Fractal geometry and the theory of chaos" is a selective discipline in the cycle of professional training for postgraduate (PhD) students (third level of education) in specialty 113- Applied Mathematics.

It is aimed at assimilating (assuring) a number of planned competencies, including the study of the modern apparatus of fractal geometry and chaos theory, as well as new methods and algorithms of mathematical physics of complex chaotic systems with possible generalizations on various classes of mathematical, physical-chemical, cybernetic, socio-economic and ecological systems, the use of modern scientific methods and the achievement of scientific results that create potentially new knowledge in the theory and practice of chaotic phenomena.

The place of discipline in the structural-logical scheme of its teaching: the acquired knowledge in the study of this discipline is used in the writing of dissertations, the topics of which are related to the study of fractal properties and the regular and chaotic dynamics of various classes of mathematical, physical and chemical, cybernetic, socio-economic and environmental systems. The basic concepts of discipline are a well-known toolkit of an experienced specialist in the field of applied mathematics.

The purpose of studying the discipline is the assimilation (assurance) of a number of competencies, in particular, the mastery of a modern apparatus of fractal geometry and chaos theory, the ability to develop new and improve existing mathematical methods of analysis, modeling and prediction on the oscillatory fractal geometry and elements of the theory of chaos of the regular and chaotic dynamics (evolution) of complex systems.

The total amount of study time involved in studying discipline is 300 hours for stationer form and 300 hours for the extramural studies.

After mastering this discipline, the postgraduate student must be able to use contemporary or develop new approaches, in particular on the basis of fractal geometry and chaos theory, to analyze, simulate, predict, and program the regular and chaotic dynamics of complex systems from the post-emerging computer experiments. These methodical instructions are for self-studying work of the second-year PhD students and tests performance in the discipline «Fractal geometry and a Chaos theory». The main topic is a theoretical studying and application of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of complex systems. As example, the problem of the : atmospheric pollutants temporal dynamics is theoretically considered and numerically studied.

Methodical instructions for self-sufficient work of PhD students, tests performance and distance learning in the discipline «Fractal geometry and a chaos theory», Part 4

Topic: Correlation integral method. Lyapunov index method. Wavelet analysis. Definition and calculation of global Lyapunov dimensions

Topic: Геометрія фазового простору. Теорія хаосу. Метод кореляційного інтегралу. Метод показників Ляпунова. Визначення та обчислення глобальних розмірностей Ляпунова ЗБ- Л3

1. Introduction

In the last years in many branches of science and technique principally new approaches to analysis and modelling dynamical system master parameters time series have become very popular. These new approaches are provided by using methods of an advanced non-linear analysis, a chaos, dynamical systems theories (c.f. [1-20] and Refs. therein).

The matter is in the fact that many processes in the Earth and environmental sciences (physics and geophysics) are nonlinear and stochastic on their nature and their studying requires using exclusively powerful mathematical methods of nonlinear analysis and a chaos and dynamical system theories. In some our previous papers [19-24] we have given a review of new methods and algorithms to analysis of different systems of quantum physics, sensor electronics and photonics and used the nonlinear method of chaos theory and the recurrence spectra formalism to study stochastic futures and chaotic elements in dynamics of physical (namely, atomic, molecular, nuclear systems in an free state and an external electromagnetic field) systems. Moreover the nontrivial manifestations of a chaos phenomenon in some very important and interesting systems have been discovered by many authors.

The authors [3,8] have presented am effective universal complex chaos-dynamical approach to the atmospheric radon ^{222}Rn concentration fluctuations analysis, modelling and prediction from beta particles activity data of radon monitors. The topological and dynamical invariants for the time series of the atmospheric ^{222}Rn concentration in the region of the Southern Finland have been calculated using the radon concentrations measurements at SMEAR II station of the Finnish Meteorological Institute.

Here we consider and present the results of application of the methods of fractal geometry and a chaos theory. Namely we presents the methodology and practical data consider computational analysis and modelling the atmospheric radon ^{222}Rn concentration temporal dynamics using the data of surface observations of the Environm. Measurement. Lab. (USA Dept. of Energy) from some sites in the United States (Chester etc).

A chaotic behaviour has been discovered and in details investigated by using nonlinear methods of the chaos and dynamical systems theories [13-18]. To reconstruct the corresponding strange chaotic attractor, the time delay and embedding dimension are computed. The former is determined by the methods of autocorrelation function and average mutual information, and the latter is calculated by means of correlation dimension method and algorithm of false nearest neighbours. The topological and dynamical invariants for the observed time series of the Rn concentrations at the Chester site are computed.

2. Chaos-geometric approach to analysis and modelling concentration time series and input data.

The time series of the atmospheric Rn concentrations extending for a least one year are available from five sites in the Unites States (Environm. Measurement. Lab., USA Dept. of Energy). The record of the radon concentrations at Chester is by far the most extensive. Measurements had been made round-the-clock 10 m above ground in a open field and data from July 1977 to November 1983 are available as continuous time series of 0.5-3 hour average concentrations (Harlee, 1978,1979; Fisenne, 1980-1985) (c.g., [2,3]. The detailed analysis of the main features for the radon data have been reviewed by Gesell and Fisenne (see [2]). The typical time series of the ^{222}Rn concentrations at Chester site (data of observations are taken from Harley,; look details in Refs. [2,3]) is presented in in Fig. 1

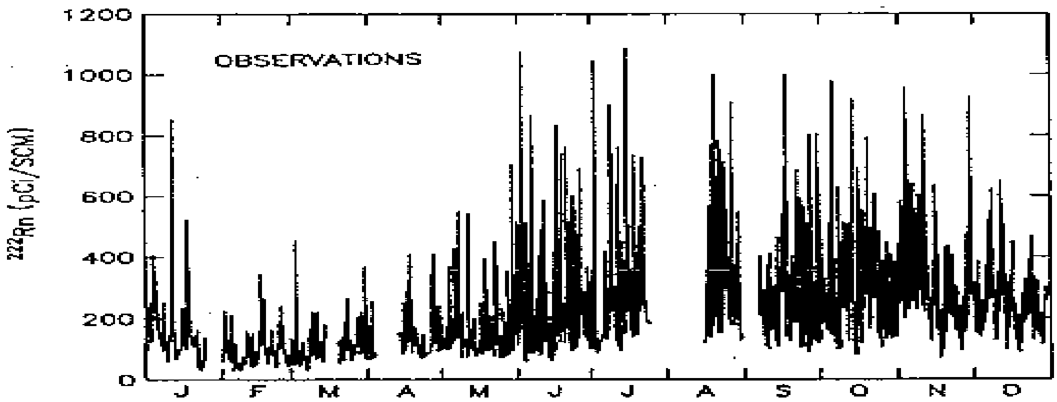

Figure 1. The typical time series of the ^{222}Rn concentrations at Chester site (data of observations) [2]).

Let us further consider the main blocks of our chaos-geometric approach, which has been presented earlier and is needed only to be reformulated regarding the problem studied in this paper. So, below we are limited only by the key moments following to Refs. [13-18].

Let us formally consider scalar measurements of the radon concentration as $s(n) = s(t_0 + n\Delta t) = s(n)$, where t_0 is the start time, Δt is the time step, and is n the number of the measurements.

Further it is necessary to reconstruct phase space using as well as possible information contained in the $s(n)$. Such a reconstruction leads to a definite set of d-dimensional vectors $\mathbf{y}(n)$ insist of initial scalar data. Further the dynamical system methods should be used. In order to reconstruct the phase space of an observed dynamical system one should apply the method of using time-delay coordinates (c.g., [13-16]).

The direct use of the lagged variables $s(n + \tau)$, where τ is some integer to be determined, results in a coordinate system in which the structure of orbits in phase space can be captured. Then using a collection of time lags to create a vector in d dimensions,

$$\mathbf{y}(n) = [s(n), s(n + \tau), s(n + 2\tau), \ldots, s(n + (d-1)\tau)],$$

the necessary required coordinates are determined. As usually, the dimension d is the embedding dimension, d_E. To determine the value of τ one should use a few methods. The first method is provided by computing the linear autocorrelation function C_L and looking for that time lag where $C_L(\delta)$ first passes through zero. The second method is provided by computing the average mutual information (look details of our version in Ref. [15]). One could remind that the autocorrelation function and average mutual information can be considered as analogues of the linear redundancy and general redundancy, respectively, which was applied in the test for nonlinearity. The general redundancies detect all dependences in the time series, while the linear redundancies are sensitive only to linear structures. Further, a possible nonlinear nature of process resulting in the vibrations amplitude level variations can be concluded.

The fundamental goal of the d_E calculation is in the further reconstruction of the Euclidean space R^d large enough so that the set of points d_A can be unfolded without ambiguity. The embedding dimension, d_E, must be greater, or at least equal, than a dimension of the corresponding chaotic attractor, d_A, i.e. $d_E > d_A$.

The correlation integral analysis is one of the widely used techniques to investigate the signatures of chaos in a time series. This method is based on using the correlation integral, $C(r)$ (c.g., [13-15]). Within this method in a case of the chaotic system the correlation exponent attains saturation with an increase in the embedding dimension. The saturation value is defined as the correlation dimension (d_2) of the attractor. The calculation of the correlation dimension can be made more exact using the method algorithm of the false nearest neighbor points.

The important step of the time series analysis is connected with computation of the Lyapunov's exponents. According to definition, the Lyapunov's exponents spectrum can be considered a measure of the effect of perturbing the initial conditions of a dynamical system. One should remember that in principle, the orbits of chaotic attractors are unpredictable, but there is the limited predictability of chaotic dynamical system, which is estimated by computing y the global and local Lyapunov's exponents. A negative values indicate local average rate of contraction while the positive values indicates a local average rate of expansion. Availability of numerical values of the Lyapunov's exponents allows easily to determine other invariants of the system such as the Kolmogorov entropy.

The inverse of the Kolmogorov entropy is equal to an average predictability. Estimate of the attractor's dimension is given by the Kaplan-Yorke conjecture:

$$d_L = j + \frac{\sum_{\alpha=1}^{j} \lambda_\alpha}{|\lambda_{j+1}|}$$

where j is such that $\sum_{\alpha=1}^{j} \lambda_\alpha > 0$ and $\sum_{\alpha=1}^{j+1} \lambda_\alpha < 0$, and the LE λ_α are taken in descending order. There are a few computational method to determine the Lyapunov's exponents. One of the wide spread methods is based on the Jacobi matrix of system. We have applied a method with linear fitted map (version [15]) , although the maps with higher order polynomials can be used too. Summing up above said and results of Refs. [13-22], a general scheme of an analysis, processing and forecasting any time series is presented in Figure 2.

I. Analytics and radioactivity dynamics

$\Downarrow$

II. Preliminary studying and conclusion regarding a chaos availibility

1. The Gottwald-Melbourne test: $K \rightarrow 1$ – chaos

$\Downarrow$

2. Energy spectrum, statistics, power spectra, ,…,

$\Downarrow$

III. The phase space geometry. The fractal geometry

3. A method of advanced autocorrelation function or average initial information algorithms

$\Downarrow$

4. Determining embedding dimension d_E by the method of the correlation dimension or algorithm of the false nearest neighbor points

$\Downarrow$

IV. Forecasting process in the environmental radioactivity dynamics

6. Computation of the global Lyapunov dimension λ_α; determination of the Kaplan-York dimension d_L (advanced algorithms)

⇓

7. Determining the number of nearest neighboring points NN for the best results,…

⇓

8. New methods and algorithms of nonlinear prediction (methods of predicted trajectories, stochastic propagators, wavelet-expansions …

Figure 2. General scheme of the non-linear analysis, modelling and sensing algorithms to compute parameters of the radioactivity dynamics time series

The "prediction" block (Figure 2) includes the methods and algorithms of nonlinear prediction such as methods of predicted trajectories, stochastic propagators, neural networks modelling, renorm-analysis with blocks of the polynomial approximations, wavelet-expansions. All calculations are performed with using "Geomath" and "Quantum Chaos" PC [15-22,27-30].

3. The results and conclusions

Table 1 summarizes the results for the time lag, which is computed for first ~10^3 values of time series. The autocorrelation function crosses 0 only for the ^{222}Rn time series, whereas this statistic for other time series remains positive. The values, where the autocorrelation function first crosses 0.1, can be chosen as τ, but earlier it had been showed that an attractor cannot be adequately reconstructed for very large values of τ. So, before making up final decision we calculate the dimension of attractor for all values in Table 1. If time lags determined by average mutual information are used, then algorithm of false nearest neighbours provides d_E = 7..

Table 1. Time lags (hours) subject to different values of C_L and first minima of average mutual information (I_{min1}) for the ^{222}Rn time series

$C_L = 0$	-
$C_L = 0.1$	258
$C_L = 0.5$	51
I_{min1}	16

Table 2 shows the results of computing a set of the dynamical and topological invariants, namely: correlation dimension (d_2), embedding dimension (d_E), two Lyapunov exponents λ_1,λ_2), Kaplan-York dimension (d_L) and average limit of predictability (Pr_{max}, hours) for the studied ^{222}Rn time series.

Table 2. The correlation dimension (d_2), embedding dimension (d_E), first two Lyapunov's
exponents, (λ_1,λ_2), Kaplan-Yorke dimension (d_L), and the Kolmogorov entropy, average
limit of predictability (Pr_{max}, hours) for the 1978 ^{222}Rn time series at the Chester site

d_2	d_E	λ_1	λ_2	K_{ent}	d_L	Pr_{max}
6,03	7	0,0194	0,0086	0,028	5,88	35

Analysis of the data shows that the Kaplan-Yorke dimensions (which are also the attractor dimensions) are smaller than the dimensions obtained by the algorithm of false nearest neighbours. It is very important to pay the attention on the presence of the two (from six) positive (chaos exists!) Lyapunov's exponents λ_i. One could conclude that the system broadens in the line of two axes and converges along four axes that in the six-dimensional space. Other values of the Lyapunov's exponents $\lambda_i$are negative.

To conclude, for the first time we have presented the results of analysis and modelling the atmospheric radon ^{222}Rn concentration time series using the data of surface observations of the Environmental Measurements Laboratory (USA Dept. of Energy) from some sites in the United States (Chester site).

We have applied such chaos and dynamical systems theories methods as autocorrelation function method and the mutual information approach, a correlation integral analysis and the false nearest neighbours algorithm, the Lyapunov exponent's analysis and surrogate data method etc. To reconstruct the corresponding strange chaotic attractor, the time delay and embedding dimension are computed. The former is determined by the methods of autocorrelation function and average mutual information, and the latter is calculated by means of correlation dimension method and algorithm of false nearest neighbours. Further, the Lyapunov's exponents spectrum, Kaplan-Yorke dimension and Kolmogorov entropy are computed. A chaotic behaviour in the atmospheric radon concentration (Chester , New Jersy) time series is firstly discovered and investigated. The Lyapunov exponent's analysis has supported this conclusion.

3. Tests performance

Test Option 1.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(Rn)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.

2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

Test Option 2.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(U)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.

2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

Test Option 3.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(Cs)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.
2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

Test Option 4.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(I)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.
2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

Test Option 5.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(Fr)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.
2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

Test Option 6.

1). Give the key definitions of of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of environmental radioactive complex systems. As example, consider radioactive pollutants temporal dynamics and formulate the definitions of the chaos-geometric approach to analysis and forecasting evolutionary dynamics of this system i) mathematical and physical essense, ii) Test by Gottwald-Melbourne, iii) spectral statistics and elements of stochasticity, iv) definitions of a chaos topological and dynamical invariants. Explain all definitions om the example of the concrete system: an environmental radioactive **(Po)** temporal dynamics from the pointwiew of fractal geometry and a chaos theory.
2). To carry out and consider realization of the numerical algorythm for computing topological and dynamical invariants. To perform its pracrical realization (using Fortran Power Station , Version 4.0; PC Code: “Superatom”) for an environmental radioactive temporal dynamics (data are preliminary taken) and calculate a chaos phenomenon topological and dynamical invariants.

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